THE

MESSENGER OF MATHEMATICS.

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EDITED BY

J. W. L. GLAISHER, Sc.D., F.R.S.,

FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

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MESSENGER OF MATHEMATICS.

ON THE NUMERICAL FACTORS OF $a^n - 1$.

(SECOND NOTICE.)

By *C. E. Bickmore, M.A.*

1. SINCE a previous communication on this subject (Vol. XXV., pp. 1–44) went to the press, another publication* has appeared, dealing with some of the same topics, viz. the tests for determining in certain cases the least solution of the congruence $10^x \equiv 1$ (mod. p), and containing as an appendix a table which gives that solution {when less than $\frac{1}{2}(p-1)$} for every prime less than 100,000.

2. In the paper itself the author—Herr Bork—gives as a generalisation of Fermat's theorem "$a^{p-1} \equiv 1$ (mod. p), when p is a prime and a is prime to p," Euler's result that a solution of the congruence $a^x \equiv 1$ (mod. N), where $N = p^\alpha p_1^\beta p_2^\gamma \ldots$, and a is prime to N, is $x = p^{\alpha-1}(p-1) \cdot p_1^{\beta-1}(p_1 - 1) \cdot p_2^{\gamma-1}(p_2 - 1) \ldots$

As already pointed out, Legendre has shown that the least common multiple of the expressions $p^{\alpha-1}(p-1)$, $p_1^{\beta-1}(p_1-1)$, $p_2^{\gamma-1}(p_2-1)$, ..., instead of their product, is an index sufficiently large.

Thus if $N = 4095 = 3^2.5.7.13$, $a^{12} \equiv 1$ (mod. N) when a is prime to N. Euler's formula would make the index 1728 instead of 12.

For particular values of a, Legendre's index may be further reduced; for if the residue-index of a for the primes $p, p_1, p_2, \ldots$ be $\lambda, \lambda_1, \lambda_2, \ldots$, respectively, $\frac{p-1}{\lambda}, \frac{p_1-1}{\lambda_1}, \frac{p_2-1}{\lambda_2}, \ldots$ should be substituted for $p-1, p_1-1, p_2-1, \ldots$; also, since $10-1=3^2$, $10^{486} \equiv 1$ (mod. 487^2), $14^{28} \equiv 1$ (mod. 29^2), $18^{36} \equiv 1$ (mod. 37^2), &c., $\alpha-1, \beta-1, \gamma-1, \ldots$ may sometimes be replaced by $\alpha-2$, $\beta-2, \gamma-2, \ldots$; Jacobi, in *Crelle's Journal*, Band III., pp. 301, 302 gives a number of cases in which $a^{p-1}-1$ is a multiple not only of the prime p, but also of p^2, p being not > 37.

* Heinrich Bork. *Periodische Dezimalbrüche.* Berlin, 1895. Programm No. 67.

3. Herr Bork also quotes what Reuschle calls an "elegant" proposition to determine the biquadratic character of 10, viz. "When $p = 4n + 1 = a^2 + b^2$, where $a \equiv 1$ (mod. 4), 10 is a biquadratic residue of p if—and only if—one of the four expressions b, $a(b \pm 4)$, $(b + a)(b - 2)$, $(b - a)(b + 2)$, be a multiple of 40."

But Reuschle's proposition is incomplete as to the last three expressions without the addition of the words, "and if 10 be a quadratic residue of p." For each of the last three expressions is respectively a multiple of 40, if $b \pm 4$, $b - 2$, or $b + 2$, be a multiple of 40; and 10 is not necessarily a quadratic or biquadratic residue of p in any of these three cases.

The condition should be stated thus:—

"When $p = 4n + 1 = a^2 + b^2$, where $a \equiv 1$ (mod. 4), $\left(\frac{10}{p}\right)_4 = 1$ in any of the following four cases, and not otherwise:

(1) when $b \equiv 0$ (mod. 40),

(2) when $a \equiv 0$ (mod. 5) and $b - 4 \equiv 0$ (mod. 8),

(3) when $a + b \equiv 0$ (mod. 5) and $b - 2 \equiv 0$ (mod. 8),

(4) when $a - b \equiv 0$ (mod. 5) and $b + 2 \equiv 0$ (mod. 8)."

In this form the test is both necessary and sufficient, and will be found to agree with the biquadratic criterion given on pp. 28–29 of my former communication.

4. The paper reproduces, from the Introduction to the *Canon Arithmeticus*, Jacobi's deductions from Burckhardt's Table as to the number of primes less than 2500 (1, 2, 5 being excluded), which have as a primitive root one or both of the bases $+10$, -10. But the errors in Burckhardt's Table make it necessary to alter Jacobi's figures.

There are not 76 primes of the form $4m + 1$ less than 2500 which have both $+10$ and -10 for primitive roots, but only 73; the residue-index of 10 for the primes 1213, 1993, 2437 is not 1, as stated by Burckhardt, but 6, 3, 2, respectively; also 1831 must be removed from the list of 73 primes of the form $4m + 3$, which have -10 as a primitive root, -10 being a cubic residue of 1831. Thus the three numbers given by Jacobi, 76, 72, 73, become 73, 72, 72, still more nearly equal to one another than the original ones. These four errors are—with three others—specially referred to in a letter written by Reuschle to Jacobi in November, 1846, and printed in Reuschle's *Abhandlung* for 1856.

Correct primitive roots for the above primes are given by Desmarest, viz. 5 for 1213, 1993, and 2437, and 3 for 1831. Jacobi's deductions from Burckhardt's incorrect figures are reproduced in Article 14 of the late Professor Henry Smith's "Report on the Theory of Numbers" (*Brit. Assoc. Report*, 1859, pp. 228–267). In this Article, consequently, for

"$148 + 73 = 221$"

should be substituted "$145 + 72 = 217$."

5. Another proposition in the 'Programm' deals with the well-known tests as to the divisibility of a number by 3 or 9 (the factors of $10^1 - 1$), or by 11 (a factor of $10^1 + 1$), and shows that they are particular cases of two more general rules as to the divisibility of a number by a factor of $10^n - 1$, where n is an odd number, or by a factor of $10^n + 1$, when n is any number, odd or even.

This was pointed out by Reuschle on p. 2 of his *Abhandlung* for 1856, and had been previously explained by Burckhardt in the Introduction to his Factor Tables.

Herr Bork applies the first rule to 41, a factor of $10^5 - 1$, and the second to 73, a factor of $10^4 + 1$.

The second rule may also be applied to correct Dr. Looff's* errors as to the factors of $10^{25} + 1$ and $10^{30} + 1$.

Dr. Looff asserts of the factor of the former,

717061202105779291,

that it is not known whether it is prime or composite; but the number may be written in the form

$$\begin{aligned}717\,(10^{15} + 1) &+ 6120\,(10^{10} - 1) + 21057\,(10^5 + 1) - 717 \\ &+ 6120 - 21057 + 79291 = (10^5 + 1)\,M + 85411 - 21774 \\ &= 11.9091.M + 7.9091.\end{aligned}$$

Thus it is a multiple of 9091, its quotient by 9091 being 78875943472201, which is the factor given by Mr. Shanks.†

Again, Dr. Looff's large factor of $10^{30} + 1$, viz.

16557360491819836049011641

may be written as

$$\begin{aligned}16557\,(10^{20} - 1) &+ (3604918198)\,(10^{10} + 1) + 16557 \\ &- 3604918198 + 3604901641 \\ &= (10^{10} + 1)\,\{16557\,(10^{10} - 1) + 3604918198\} + 0 \\ &= (10^{10} + 1)\,(165573604901641).\end{aligned}$$

* Grunert, *Archiv.* 1851., p. 54. † See note on p. 4.

Thus it should be divided by $10^{10} + 1$, not—as incorrectly stated in the former portion of this paper—by

$$(10^{10} + 1) \div (10^2 + 1).$$

The quotient is the result given by Mr. Shanks, and—as already shown—is the product of the two primes 4188901 and 39527641.

6. The table which constitutes the appendix, calculated by Dr. Kessler, of Wiesbaden, adds ten factors to those given by Mr. Shanks in his table of factors* of $10^n - 1$ when n is not greater than 100.

The following table gives the factors, all but one of which (viz. 98641) are included in Dr. Salmon's unpublished calculations.

n	29	55	63	72	77	87	95	100
Factors	43037×62003	62921	45613	98641	42063	72559	59281×63841	60101

The last result shows that one of the two large factors of $(10^{50} + 1) \div (10^{10} + 1)$, given on p. 34 of the former part of this paper, is a multiple of 60101; and in fact the larger one is the multiple.

For $60101 = 95^2 + 226^2$; and

$$(10049999499^2 + 50300500^2) \div (95^2 + 226^2)$$
$$= (16074905^2 + 37711874^2).$$

Hence

$$(10^{50} + 1) \div (10^{10} + 1)$$
$$= 60101.1680588011350901.990049800698004990 01.$$

7. Besides the error pointed out in (5), and the additional factors referred to in (6), there are other corrections and additions required in the former portion of this paper. A table of addenda and corrigenda is given at the end of this portion; the following paragraphs supply further additions.

8. The algebraical prime-factor of $a^{15} - b^{15}$ can be expressed in a form specially suited to the case when either $3ab$ or $-3ab$ is a square number, *i.e.* when $a = 3x^2$ and $b = \pm y^2$.

* *Proc. Roy. Soc.*, Vol. XXII., pp. 381–384.

For

$$\frac{(a^{15}-b^{15})(a-b)}{(a^5-b^5)(a^3-b^3)}=\frac{(a^5-b^5)^2+3a^5b^5}{(a-b)^2+3ab}$$

$$=(a^4-2a^3b+a^2b^2-2ab^3+b^4)^2+3ab(a^3-a^2b+ab^2-b^3)^2;$$

now, if $a=3x^2$ and $b=y^2$, this becomes

$$\frac{(3^{15}x^{30}-y^{30})(3x^2-y^2)}{(3^5x^{10}-y^{10})(3^3x^6-y^6)}=(3^4x^8-2.3^3x^6y^2+3^2x^4y^4-2.3x^2y^6+y^8)^2$$
$$+(3^4x^7y-3^3x^5y^3+3^2x^3y^5-3xy^7)^2;$$

again, if $a=3x^2$ and $b=-y^2$, $\dfrac{(3^{15}x^{30}+y^{30})(3x^2+y^2)}{(3^5x^{10}+y^{10})(3^3x^6+y^6)}=$ the difference of two squares.

Thus the algebraical prime-factor of $3^{15}-1$, $3^{45}-1$, $3^{75}-1$, $3^{135}-1$, $12^{15}-1$, $12^{45}-1$, $12^{75}-1$, $12^{135}-1$ is expressed as the sum of two squares; and the algebraical prime-factor of $3^{15}+1$, $3^{45}+1$, $3^{75}+1$, $3^{135}+1$, $12^{15}+1$, $12^{45}+1$, $12^{75}+1$, $12^{135}+1$ as the product of two numerical factors.

9. In my former communication, no direct criterion was given for the cubic character of 6, in the case when 2 and 3 are each a cubic non-residue of a prime of the form $6m+1$; the following gives the criterion for the cubic character of 6, and also of 12.

When $4p=L^2+27M^2$, according to the form of the remainder when L and M are divided by 6, either each of the four bases 2, 3, 6, 12 is a cubic residue of p, or one of the four is a cubic residue and each of the other three is a cubic non-residue.

The necessary and sufficient conditions are:

(1) If $M\equiv 0$ (mod. 6), each of the four is a cubic residue.

(2) If $M\equiv \pm 2$ (mod. 6), 2 is the only cubic residue of the four.

(3) If $M\equiv 3$ (mod. 6), 3 is the only cubic residue.

(4) If $L\equiv M\equiv 1$ (mod. 4),* and $L-M\equiv 0$ (mod. 6), 6 is the only cubic residue.

(5) If $L\equiv M\equiv 1$ (mod. 4),* and $L+M\equiv 0$ (mod. 6), 12 is the only cubic residue.

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* These congruences fix the sign of the uneven numbers L, M, which must be taken negative, if the numerical value be of the form $4m-1$.

As examples of case (4), any factor of $(6^{12n\pm2} + 1)$ may be taken: for

$$6^{12n+2} + 1 = (6^{6n+1} - 1)^2 + 3.(2.6^{3n})^2 = (6^{6n+1} + 1)^2 - 3.(2.6^{3n})^2,$$

and

$$6^{12n-2} + 1 = (6^{6n-1} - 1)^2 + 3.(2.6^{3n-1})^2 = (6^{6n+1} - 1)^2 - 3.(2.6^{3n})^2.$$

Hence, any proper prime-factor is of the form $12(6n\pm1)M+1$, with 6 for a residue of order $3M$, *i.e.* 6 is a cubic residue of every proper prime factor.

Thus $(6^{14} + 1) \div (6^2 + 1) = 421.5030761$, and 6 should satisfy the criterion (4) for the prime 5030761, which is equal to

$$2069^2 + 3.500^2 = \tfrac{1}{4}(3569^2 + 27.523^2).$$

Therefore $L = 3569$, $M = -523$, $L - M = 4092 \equiv 0 \pmod{6}$.

As an example of case (5), the table gives $12^{42} \equiv 1$ (mod. 17011), so that 12 is a residue of 17011 of order 405.

Now $\quad 17011 = 32^2 + 3.73^2 = \tfrac{1}{4}(187^2 + 27.35^2)$;

therefore $\quad L = -187, \quad M = -35,$

$$L + M = -222 \equiv 0 \pmod{6}.$$

Thus, the criterion shows that 6 is a cubic residue of 5030761, and 12 a cubic residue of 17011, a result agreeing with what has been proved by other methods.

10. The biquadratic character of 6 when $p = 24m + 5$ was also not completely determined: the criterion is very similar to the biquadratic criterion for 10, and may be stated thus:

"When $p = 4n + 1 = a^2 + b^2$, where $a \equiv 1 \pmod{4}$,

$\left(\dfrac{6}{p}\right)_4 = 1$, in any of the four following cases, and not otherwise;

(1) when $b \equiv 0 \pmod{12}$,

(2) when $a \equiv 0 \pmod{3}$, and $b - 4 \equiv 0 \pmod{8}$,

(3) when $a + b \equiv 0 \pmod{3}$, and $b + 2 \equiv 0 \pmod{8}$,

(4) when $a - b \equiv 0 \pmod{3}$, and $b + 2 \equiv 0 \pmod{8}$."

As an example of (3), one may take $p = 173 = 13^2 + 2^2$; therefore $a + b = 15 \equiv 0 \pmod{3}$, $b - 2 = 0 \equiv 0 \pmod{8}$; therefore $\left(\dfrac{6}{173}\right)_4 = 1$, *i.e.* 173 is a factor of $6^{43} - 1$, a result obtainable by the Canon Arithmeticus.

For case (4), if $p = 1013 = 23^2 + 22^2$, $a = -23$, $b = 22$;

therefore
$$a - b = -45 \equiv 0 \pmod{3},$$
$$b + 2 = \quad 24 \equiv 0 \pmod{8};$$

therefore $\left(\dfrac{6}{1013}\right)_4 = 1$, *i.e.* 1013 is a factor of $6^{253} - 1$.

Both for case (3) and case (4), it may be noted that ab is not a multiple of 3, but that $a^2 - b^2$ is.

Gauss (*Werke*, Band II., p. 101) gives other forms of the biquadratic criterion for 6, which will be found to agree with the above.

The conditions that 6 should be a cubic or biquadratic residue (and also those for 2, 3, 5, 7, 10) were determined conjecturally by Euler, and will be found in Vol. I. of Euler's posthumous works (published by his great grandsons, P. and N. Fuss), as part of an unfinished Treatise on the *Theory of Numbers.*

11. The resolution for $3^{45} - 1$ given on page 42 is incorrect and incomplete.

It should have been as follows: (see p. 5)

$$\frac{(3^{45}-1)(3^3-1)}{(3^{15}-1)(3^9-1)} = (3^{12} - 2.3^9 + 3^6 - 2.3^3 + 1)^2 + (3^{11} - 3^8 + 3^5 - 3^2)^2$$
$$= 492751^2 + 170820^2 = (9^2 + 10^2)(33939^2 + 18730^2)$$
$$= (9^2 + 10^2)(39^2 + 10^2)(701^2 + 660^2) = 181.1621.927001.$$

The expression of the three factors in the form $A^2 + 3B^2$ is obtainable from another identity, applicable when ab is an uneven number, viz.

$$\frac{(a^{15} - b^{15})(a - b)}{(a^5 - b^5)(a^3 - b^3)} = \left(\frac{a^4 + a^3b - 2a^2b^2 + ab^3 + b^4}{2}\right)^2$$
$$+ 3\left(\frac{a^4 - a^3b + ab^3 - b^4}{2}\right)^2;$$

whence, putting $a = 3^3$, $b = 1$, the algebraical prime-factor of $3^{45} - 1$

$$= 274847^2 + 3.255892^2 = (13^2 + 3.2^2)(28223^2 + 3.15342^2)$$
$$= (13^2 + 3.2^2)(13^2 + 3.22^2)(851^2 + 3.260^2).$$

Thus the biquadratic character and cubic character of different bases for the prime 927001 are deducible, since

$$927001 = 701^2 + 660^2 = 851^2 + 3.260^2.$$

The factor 1621 was obtained from Col. Cunningham's result $2^{738}+3 \equiv 0 \pmod{1621}$; from another result in his Canon, viz. $2^{55} \equiv 3 \pmod{1871}$, one obtains $\left(\frac{3}{1871}\right)_{110} = 1$, *i.e.* $3^{17} \equiv 1 \pmod{1871}$.

By actual division $3^{17}-1 = 2.1871.34511$.

Thus two more numbers in the column* of 3^n-1 are completely resolved, viz. $3^{17}-1$ and $3^{45}-1$.

12. Identities similar to those given for base 10 on p. 33 may be found for base 6.

For, if $6ab =$ a square $= 36R^2$, then $\sqrt{(6ab)} = 6R$, a condition which is satisfied by putting $R = xy$, and either

$$(\alpha)\ a = 6x^2,\ b = y^2 \ \text{ or }\ (\beta)\ a = 2x^2,\ b = 3y^2.$$

In this case

$$(1)\ \frac{a^6+b^6}{a^2+b^2} = \{(a-3R)^2 + (3R-b)^2\}\{(a+3R)^2 + (3R+b)^2\}$$

$$= \{(\tfrac{1}{2}a+b-3R)^2 + 3(\tfrac{1}{2}a-R)^2\}\{(\tfrac{1}{2}a+b+3R)^2 + 3(\tfrac{1}{2}a+R)^2\}$$

$$= \{(a+b-3R)^2 - 3R^2\}\{(a+b+3R)^2 - 3R^2\},$$

$$(2)\ \frac{(a^{30}+b^{30})(a^2+b^2)}{(a^{10}+b^{10})(a^6+b^6)}$$

$$= [\{(a^4-a^2b^2+b^4) - 3R(a^3-a^2b-ab^2+b^3)\}^2$$
$$+ \{3R(a^3+a^2b-ab^2-b^3) - (2a^3b-2ab^3)\}^2]$$

$$\times [\{(a^4-a^2b^2+b^4) + 3R(a^3-a^2b-ab^2+b^3)\}^2$$
$$+ \{3R(a^3+a^2b-ab^2-b^3) + (2a^3b-2ab^3)\}^2]$$

$$= [\{(\tfrac{1}{2}a^4+a^3b-\tfrac{1}{2}a^2b^2-\tfrac{1}{2}ab^3-b^4) - 3R(a^3-b^3)\}^2$$
$$+ 3\{R(a^3-b^3) - (\tfrac{1}{2}a^4+\tfrac{1}{2}a^2b^2-\tfrac{1}{2}ab^3)\}^2]$$

$$\times [\{(\tfrac{1}{2}a^4+a^3b-\tfrac{1}{2}a^2b^2-\tfrac{1}{2}ab^3-b^4) + 3R(a^3-b^3)\}^2$$
$$+ 3\{R(a^3-b^3) + (\tfrac{1}{2}a^4+\tfrac{1}{2}a^2b^2-\tfrac{1}{2}ab^3)\}^2].$$

* *Messenger*, Vol. XXV., p. 43.

If in (1) $a = 6^3$, $b = 1$, then $R = 6$, and the result is

$$(3)\ \frac{6^{18}+1}{6^6+1} = (17^2 + 198^2)(19^2 + 234^2)$$

$$= (3^2 + 8^2)(21^2 + 10^2)(19^2 + 234^2)$$

$$= (91^2+3.102^2)(127^2+3.114^2) = (5^2+3.4^2)(23^2+3.2^2)(127^2+3.114^2)$$

$$= (199^2 - 3.6^2)(235^2 - 3.6^2) = (10^2 - 3.3^2)(28^2 - 3.9^2)(235^2 - 3.6^2)$$

$$= 73.541.55117.$$

If in (2) $a = 6$, $b = 1$, then $R = 1$, and the result is

$$(4)\ \frac{(6^{30}+1)(6^2+1)}{(6^{10}+1)(6^6+1)} = (315^2 + 736^2)(1155^2 + 1786^2)$$

$$= (9^2 + 10^2)(25^2 + 54^2)(5^2 + 6^2)(81^2 + 260^2)$$

$$= (197^2 + 3.448^2)(1487^2 + 3.878^2)$$

$$= (13^2 + 3.2^2)(29^2 + 3.30^2)(7^2 + 3.2^2)(257^2 + 3.52^2)$$

$$= 181.3541.61.74161.$$

13. The tests for the cubic and biquadratic character of 11 were referred to on p. 32, but not stated.

The cubic test is, that when $p = 6m+1$, so that $4p = L^2 + 27M^2$, $\left(\frac{11}{p}\right)_3 = 1$, if—and only if—one of the four numbers L, M, $L + 4M$, $L - 4M$, be a multiple of 11.

The biquadratic test is, that when $p = 4m+1$, so that $p = a^2 + b^2$, $\left(\frac{\overline{11}}{p}\right)_4 = 1$, if—and only if—one of the three numbers b, $a+2b$, $a-2b$, be a multiple of 11.

The laws of reciprocity from which they are deduced may be thus enunciated. "It is possible to solve the congruence $x^3 \equiv q$ (mod. p), where $p = 6m + 1 = \frac{1}{4}(L^2 + 27M^2)$, and q is a prime greater than 3, if—and only if—it is possible to solve the congruence $y^3 \equiv \dfrac{L + M\sqrt{(-27)}}{L - M\sqrt{(-27)}}$ (mod. q)," and "It is possible to solve the congruence $x^4 \equiv q$ (mod. p), where

$p = 4m + 1 = a^2 + b^2$, and q is any uneven prime, if—and only if—it is possible to solve the congruence

$$y^4 \equiv \frac{a + b\sqrt{(-1)}}{a - b\sqrt{(-1)}} \text{ (mod. } q).\text{"}$$

Hence

$$\text{"}\left(\frac{q}{p}\right)_3 = 1, \text{ if—and only if—} \left(\frac{L + M\sqrt{(-27)}}{L - M\sqrt{(-27)}}\right)^{\frac{1}{3}(q-1)} \equiv 1 \text{ (mod. } q),\text{"}$$

and

$$\text{"}\left(\frac{q}{p}\right)_4 = 1, \text{ if— and only if—} \left(\frac{a + b\sqrt{(-1)}}{a - b\sqrt{(-1)}}\right)^{\frac{1}{4}(q-1)} \equiv 1 \text{ (mod } q).\text{"}$$

The sign given to q must be such as to make $\frac{1}{3}(q-1)$, $\frac{1}{4}(q-1)$ an integer, positive or negative: thus, if $q = \pm 31$, the positive sign must be taken to find the cubic character, and the negative sign to find the biquadratic character.

In fact, $\left(\frac{31}{p}\right)_3 = 1$, if—and only if—

$$\{L + M\sqrt{(-27)}\}^5 \pm \{L - M\sqrt{(-27)}\}^5 \equiv 0 \text{ (mod. 31)},$$

and $\left(\frac{\overline{31}}{p}\right)_4 = 1$, if—and only if—

$$\{a + b\sqrt{(-1)}\}^4 \pm \{a - b\sqrt{(-1)}\}^4 \equiv 0 \text{ (mod. 31)}.$$

14. These laws of cubic and biquadratic reciprocity (of which the rigorous proof was first given by Eisenstein), depend on the theory of the resolution of a prime into imaginary factors involving the roots of 1, *i.e.* the theory of complex integers.

The following is a short account of that theory.

A complex integer of order n is an expression of the form

$$a_0 + a_1\rho + a_2\rho^2 + \ldots + a_{n-1}\rho^{n-1} = f(\rho),$$

where $a_0, a_1, a_2, \ldots, a_{n-1}$ are real integers, and $1, \rho, \rho^2, \ldots, \rho^{n-1}$ are the n roots of $x^n = 1$.

If $n = 1$ or 2, $f(\rho)$ is a real integer.

If $n = 3$, $f(\rho) = a_0 + a_1\rho + a_2\rho^2$, where $\rho^2 + \rho + 1 = 0$.

If $n = 4$, $f(\rho) = a_0 + a_1\rho + a_2\rho^2 + a_3\rho^3$, where $\rho^2 + 1 = 0$.

The product $f(\rho_1) . f(\rho_2) . f(\rho_3) \ldots$, where $\rho_1, \rho_2, \rho_3, \ldots$ are those n^{th} roots of 1 which are not m^{th} roots, m being a submultiple of n, is called the Norm of the complex integer, and is denoted by the symbol $Nf(\rho)$, or $N_n f(\rho)$, n being the order of the integer.

The theorems

$$p = 4M + 1 = a^2 + b^2,$$

$$p = 6M + 1 = A^2 + 3B^2,$$

which were enunciated by Fermat in a letter to Sir Kenelm Digby written in the year 1658, are examples of the theorem (which holds for any prime value of λ less than 23, and also when λ is any power of 2) "Every prime of the form $2\lambda M + 1$ is the norm of a complex integer of order 2λ." Complex integers of order λ, when λ is an odd number, are at once transformed to those of order 2λ by writing $-\rho$ for ρ.

Thus, if $\lambda = 2$, $f(\rho)$ is a biquadratic integer, and is equal to

$$a_0 + a_1\sqrt{(-1)} - a_2 - a_3\sqrt{(-1)} = (a_0 - a_2) + (a_1 - a_3)\sqrt{(-1)},$$

i.e. to $a + b\sqrt{(-1)}$; and

$$p = \{a + b\sqrt{(-1)}\}\{a - b\sqrt{(-1)}\} = a^2 + b^2 = N_4\{a + b\sqrt{(-1)}\}.$$

If $\lambda = 3$, $f(\rho)$ is a sextic (or cubic) integer, and is equal to

$$a_0 + a_1\rho + a_2\rho^2,$$

which may be written in the forms

$$(a_0 - a_2) + (a_1 - a_2)\rho, \quad (a_0 - a_1) + (a_2 - a_1)\rho^2;$$

therefore

$$p = (a_0 + a_1\rho + a_2\rho^2)(a_0 + a_1\rho^2 + a_2\rho)$$
$$= (a_0 - a_1)^2 + (a_0 - a_1)(a_1 - a_2) + (a_1 - a_2)^2,$$

which can be expressed in the form $A^2 + 3B^2$ in three ways, according as $a_0 - a_1$, $a_1 - a_2$, or $a_0 - a_2$, is an even number.

Conversely,

$$A^2 + 3B^2 = \{A + B\sqrt{(-3)}\}\{A - B\sqrt{(-3)}\}$$
$$= \{A + B(\rho - \rho^2)\}\{A - B(\rho - \rho^2)\}$$

$$= \text{the norm of } A + B\rho - B\rho^2 \text{ or } A - B\rho + B\rho^2,$$

a cubic (or sextic) integer.

It does not follow conversely that the norm of a complex integer of order 2λ is necessarily a prime of the form $2\lambda m + 1$.

Thus $\quad N_4(1-\rho) = N_4(1+\rho) = (1-\rho)(1+\rho) = 2,$

and $$N_3(1-\rho) = (1-\rho)(1-\rho^2) = 3.$$

These resolutions of 2 into biquadratic integers, and of 3 into cubic or sextic integers, furnish direct and simple proofs of the rules for the quadratic character of 2* and of 3, suggested by Fermat, and proved by Lagrange and Euler respectively.

Thus, if $\qquad i^2 = -1, \quad 2 = (1+i)(1-i);$

therefore $2i = (1+i)^2$; raising each member to the power $\frac{1}{2}(p-1)$,
$$2^{\frac{1}{2}(p-1)} i^{\frac{1}{2}(p-1)} = (1+i)^{p-1}.$$

Therefore, multiplying each member by $(1+i)$,
$$\begin{aligned} 2^{\frac{1}{2}(p-1)}\left(i^{\frac{1}{2}(p-1)} + i^{\frac{1}{2}(p+1)}\right) &= (1+i)^p \\ &= \{c_0 - c_2 + c_4 - \ldots + (-1)^{\frac{1}{2}(p-1)} c_{p-1}\} \\ &\quad + \{c_1 - c_3 + c_5 - \ldots + (-1)^{\frac{1}{2}(p-1)} c_p\}\, i, \end{aligned}$$

c_n being the coefficient of x^n in the expansion of $(1+x)^p$.

Now $c_0 = c_p = 1$; and, p being a prime, any value of c_n other than c_0 or c_p is a multiple of p.

Thus, equating real parts, if $\frac{1}{2}(p-1)$ be even,
$$2^{\frac{1}{2}(p-1)}(-1)^{\frac{1}{4}(p-1)} = 1 + \text{a multiple of } p;$$

and, if $\frac{1}{2}(p+1)$ be even,
$$2^{\frac{1}{2}(p-1)}(-1)^{\frac{1}{4}(p+1)} = 1 + \text{a multiple of } p,$$

whence $$2^{\frac{1}{2}(p-1)} \equiv (-1)^{\frac{1}{2}\cdot\frac{1}{2}(p-1)\cdot\frac{1}{2}(p+1)} \pmod{p}.$$

Similarly, if $\quad \rho^2 + \rho + 1 = 0, \;\; 3 = (1-\rho)(1-\rho^2),$

therefore $\quad -3\rho = (1-\rho)^2, \;\; (-3)^{\frac{1}{2}(p-1)} \rho^{\frac{1}{2}(p-1)} = (1-\rho)^{p-1},$
$$(-3)^{\frac{1}{2}(p-1)} \rho^{\frac{1}{2}(p-1)} (1-\rho) = (1-\rho)^p,$$

i.e.

$$(-3)^{\frac{1}{2}(p-1)}\left(\rho^{\frac{1}{2}(p-1)} - \rho^{\frac{1}{2}(p+1)}\right)$$
$$= (c_0 - c_3 + c_6 - \ldots) - (c_1 - c_4 + c_7 - \ldots)\rho + (c_2 - c_5 + c_8 - \ldots)\rho^2.$$

---

* Cauchy, *Théorie des Nombres*, p. 451.

Now, p being a prime greater than 3, either $\frac{1}{2}(p-1)$ or $\frac{1}{2}(p+1)$ is a multiple of 3; and, if $\frac{1}{2}(p-1)$ be a multiple of 3,

$$(-3)^{\frac{1}{2}(p-1)} \equiv (-1)^{\frac{1}{6}(p-1)} \pmod{p};$$

also, if $\frac{1}{2}(p+1)$ be a multiple of 3,

$$-(-3)^{\frac{1}{2}(p-1)} \equiv (-1)^{\frac{1}{6}(p+1)} \pmod{p}.$$

Jacobi* has specially considered the cases $p = 2\lambda m + 1$, where $\lambda = 4$, 5, 6, values of λ, for which p is the norm of a $2\lambda^{ic}$ integer.

Thus, if $\lambda = 4$,

$$f(\rho) = a_0 + a_1\rho + a_2\rho^2 + a_3\rho^3 + a_4\rho^4 + a_5\rho^5 + a_6\rho^6 + a_7\rho^7,$$

where $\qquad \rho^4 + 1 = 0$, *i.e.* $\pm\rho\sqrt{2} = 1 \pm \sqrt{(-1)}$,

hence

$$f(\rho) = (a_0 - a_4) + (a_1 - a_5)\rho + (a_2 - a_6)\rho^2 + (a_3 - a_7)\rho^3$$
$$= \alpha + \beta\rho + \gamma\rho^2 + \delta\rho^3.$$

If the upper signs be taken in $\pm\rho\sqrt{2} = 1 \pm \sqrt{(-1)}$,

$$\rho\sqrt{2} = 1 + \sqrt{(-1)}, \quad \rho^2 = \sqrt{(-1)},$$
$$\rho^3\sqrt{2} = -1 + \sqrt{(-1)}, \quad \rho^5 = -\rho, \quad \rho^7 = -\rho^3,$$

therefore $\quad f(\rho) = \{\alpha + \gamma\sqrt{(-1)}\} + \{\beta + \delta\sqrt{(-1)}\}$,

$$f(\rho^3) = \{\alpha - \gamma\sqrt{(-1)}\} + \rho\,\{\beta\sqrt{(-1)} + \delta\},$$
$$f(\rho^5) = \{\alpha + \gamma\sqrt{(-1)}\} - \rho\,\{\beta + \delta\sqrt{(-1)}\},$$
$$f(\rho^7) = \{\alpha - \gamma\sqrt{(-1)}\} - \rho\,\{\beta\sqrt{(-1)} + \delta\},$$

therefore

$$f(\rho)\cdot f(\rho^5) = \{\alpha + \gamma\sqrt{(-1)}\}^2 - \sqrt{(-1)}\,\{\beta + \delta\sqrt{(-1)}\}^2$$
$$= (\alpha^2 + 2\beta\delta - \gamma^2) + (2\alpha\gamma - \beta^2 + \delta^2)\sqrt{(-1)}$$
$$= a + b\sqrt{(-1)},$$

$$f(\rho^3)\cdot f(\rho^7) = \{\alpha - \gamma\sqrt{(-1)^2}\} - \sqrt{(-1)}\,\{\beta\sqrt{(-1)} + \delta\}^2$$
$$= (\alpha^2 + 2\beta\delta - \gamma^2) - (2\alpha\gamma - \beta^2 + \delta^2)\sqrt{(-1)}$$
$$= a - b\sqrt{(-1)},$$

and $p = N_8 f(\rho) = a^2 + b^2$.

---

* *Crelle*, Band XIX., pp. 314–318, or Liouville VIII. (1843), pp. 268–272.

Also

$$f(\rho).f(\rho^3) = (\alpha^2 - \beta^2 + \gamma^2 - \delta^2) + \rho\{1 + \sqrt{(-1)}\}\{\alpha(\beta + \delta) - \gamma(\beta - \delta)\}$$
$$= c + d\sqrt{(-2)},$$

since $$\rho\{1 + \sqrt{(-1)}\} = \sqrt{(-2)}.$$

Hence $$f(\rho).f(\rho^7) = c - d\sqrt{(-2)},$$

and $$p = N_8 f(\rho) = c^2 + 2d^2.$$

The result $$\text{“}p = 8m + 1 = c^2 + 2d^2\text{”}$$

was also enunciated in Fermat's letter above quoted.

A third way of expressing p in a quadratic form was added by Lagrange, and is found by noting that

$$f(\rho).f(\rho^7) = (\alpha^2 + \beta^2 + \gamma^2 + \delta^2) + \rho\{1 - \sqrt{(-1)}\}\{\alpha(\beta - \delta) + \gamma(\beta + \delta)\},$$

which, since $\rho\{1 - \sqrt{(-1)}\} = \sqrt{2}$, is equal to $e + f\sqrt{2}$;

therefore $$p = N_8 f(\rho) = e^2 - 2f^2.$$

Thus a, b, c, d, e, f are definite quadratic functions of $\alpha, \beta, \gamma, \delta$; and there are six equations, reducible to four by the two identities

$$a^2 + b^2 = c^2 + 2d^2 = e^2 - 2f^2,$$

to determine $\alpha, \beta, \gamma, \delta$, when a, b, c, d, e, f are known.

a, b have been determined by Zornow for all primes of the form $8m + 1$ less than 12000 and c, d by Jacobi for every such prime to 5953 inclusive.*

There is only one set of values of a, b, c, d, for any such prime; but there are an infinite number of ways of expressing p in the form $e^2 - 2f^2$: to determine the $\alpha, \beta, \gamma, \delta$ so as to make α odd, β, γ, δ even, and $\alpha + \beta + \gamma + \delta \equiv 1$ (mod. 4), it is convenient to take for e the least value of the form $4k + 1$, for which $p = e^2 - 2f^2$.

a, c, e are necessarily odd numbers, and $\frac{1}{2}b, d, f$ necessarily even numbers, and e is positive.

---

* See tables in *Crelle's Journal*, Band XXX., pp. 174–180, and extensions in Reuschle's *Math. Abhandl.* 1856.

The signs of a, c and the value of e are taken, so that

$$a \equiv c \equiv e \equiv 1 \pmod{4}.$$

If $ab \equiv 0 \pmod{3}$, $d \equiv 0 \pmod{3}$; and if $a^2 - b^2 \equiv 0 \pmod{3}$, $c \equiv 0 \pmod{3}$.

15. From consideration of the results in the preceding paragraph, Jacobi seems to have deduced criteria as to the octavic character of a base, *i.e.* as to whether $x^8 \equiv r \pmod{p}$, when $p \equiv 1 \pmod{8}$, does, or does not, admit of eight solutions between 0 and p.

The only results published are for the cases of $r=2$, $r=10$; the latter is given (without proof) in Jacobi's letter to Reuschle, of December, 1846; it necessarily involves the criterion for $r=5$, and that for $r=2$, (subsequently given by Reuschle), the latter depending on a and b only, and not, like those for other values of r, on c and d also.

The following criteria rest on induction from a very large number of cases, that for 5 depending also on Jacobi's theorem as to the octavic character of 10, which seems to have suggested to Reuschle his octavic criterion for 2.

16. (1) Octavic character of $\bar{3}$ and 3.

If $p = 8m+1 = a^2+b^2$, $b \equiv 0 \pmod{4}$, and $\left(\frac{\bar{3}}{p}\right)_4 = 1$ if—and only if—$b \equiv 0 \pmod{3}$; which gives [as does also $a \equiv 0 \pmod{3}$], $m \equiv 0 \pmod{3}$, and [since $p = c^2 + 2d^2$,] $d \equiv 0 \pmod{3}$.

Hence $a \equiv c \equiv 1 \pmod{4}$, $\frac{1}{2}b \equiv d \equiv 0 \pmod{6}$, are preliminary conditions, if $\left(\frac{\bar{3}}{p}\right)_4 = 1$ and $\left(\frac{\bar{3}}{p}\right)_8 = \pm 1$.

When these are satisfied, $\left(\frac{\bar{3}}{p}\right)_8 \equiv ac \pmod{3}$;

and since $\left(\frac{\bar{3}}{p}\right)_8 = (-1)^{\frac{1}{8}(p-1)} \left(\frac{3}{p}\right)_8$,

$$\left(\frac{3}{p}\right)_8 \equiv (-1)^{\frac{1}{8}(p-1)} ac \pmod{3}.$$

Thus, if $p = 8209 = \overline{55}^2 + 72^2 = 89^2 + 2.12^2$, the preliminary conditions are satisfied, and

$$\left(\frac{3}{p}\right)_8 \equiv (-1)^{1026} \times \overline{55} \times 89 \pmod{3} = 1.$$

Thus $3^{1026} - 1$ is a multiple of 8209.

8209 is known to be a factor of $3^{27} - 1$, and 27 is a submultiple of 1026; thus in fact $\left(\frac{3}{8209}\right)_{304} = 1$, which agrees with the result obtained by the criterion, viz. $\left(\frac{3}{8209}\right)_8 = 1$.

Again, if $p = 2521$, $a = \overline{35}$, $b = 36$, $c = 37$, $d = 24$, and the preliminary conditions are satisfied.

The criterion gives

$$\left(\frac{\overline{3}}{2521}\right)_8 \equiv \overline{35} \times 37 \text{ (mod. 3)} = 1,$$

therefore $$\left(\frac{3}{2521}\right)_8 = (-1)^{315} = -1,$$

and 2521 is a factor of $3^{315} + 1$.

Now resolving $3^{63} + 1$ by Aurifeuille's process, one factor is $\frac{3^{21} - 3^{11} + 1}{3^3 + 3^2 + 1}$, which is the product of the three primes 127, 883, 2521.* Since 63 is a submultiple of 315, this is another verification of the criterion.

(2) Octavic character of $\overline{6}$ and 6.

This has to be found when $p = 8m + 1$, and $\left(\frac{\overline{6}}{p}\right)_4 = \left(\frac{6}{p}\right)_4 = 1$, the latter conditions being satisfied in two ways, as follows:

(α) $\left(\frac{2}{p}\right)_4 = \left(\frac{3}{p}\right)_4 = 1$.

The necessary and sufficient conditions for this are

$$b \equiv 0 \text{ (mod. 24)}, \quad c \equiv \pm 1 \text{ (mod. 6)}, \quad d \equiv 0 \text{ (mod. 6)},$$

the two latter being included in the first.

In this case $\left(\frac{\overline{6}}{p}\right)_8 = \left(\frac{2}{p}\right)_8 \times \left(\frac{\overline{3}}{p}\right)_8 \equiv (-1)^{\frac{1}{4}(a-1)+\frac{1}{8}b} \times ac \text{ (mod. 3)}$.

(β) $\left(\frac{2}{p}\right)_4 = \left(\frac{3}{p}\right)_4 = -1$.

---

* The part that $\left(\frac{\overline{3}}{2521}\right)_8 = 1$ may be shewn by eliminating 35 between $2521 = 35^2 + 36^2 = 1 + 2.35.36$, the result being $2^{10}.3^8 \equiv -1$ (mod. 2521).

CONTENTS.

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$$x = 2,\ y = 1 \text{ gives } \frac{(12^{15}+1)(12+1)}{(12^3+1)(12^5+1)}$$

$$= N_5(1 - 6\rho + 12\rho^2) \times N_5(1 + 6\rho + 12\rho^2).$$

Now

$$(1 - 6\rho + 12\rho^2)(1 - 6\rho^4 + 12\rho^3) = 181 - 78(\rho + \rho^4) + 12(\rho^2 + \rho^3),$$

and

$$(1 + 6\rho + 12\rho^2)(1 + 6\rho^4 + 12\rho^3) = 181 + 78(\rho + \rho^4) + 12(\rho^2 + \rho^3),$$

also $\rho + \rho^4 = \frac{1}{2}\surd(5 - 1)$, $\rho^2 + \rho^3 = -\frac{1}{2}\surd(5 + 1)$;

therefore

$$N_5(1 - 6\rho + 12\rho^2) = 214^2 - 5.45^2 = 35671 = \text{a prime},$$

and

$$N_5(1 + 6\rho + 12\rho^2) = 136^2 - 5.33^2 = 13051 = 31.421.$$

Now $\rho(1 - 6\rho + 12\rho^2) = \rho - 6\rho^2 + 12\rho^3$ is a 'primary' quintic factor of 35671 satisfying both conditions, and

$$\tfrac{1}{5}(a_1 - a_4) + \tfrac{2}{5}(a_2 - a_3) = \tfrac{1}{5}(1 - 126) \equiv 0 \pmod{5},$$

therefore

$$\left(\frac{5}{35671}\right)_5 = \left(\frac{5}{35671}\right)_{10} = \left(\frac{5}{35671}\right)_{30} = 1,$$

the cubic condition being satisfied, since

$$35671 = \tfrac{1}{4}(247^2 + 27.55^2).$$

Thus 35671 is a factor of $5^{1189} - 1$, a multiple of $5^{29} - 1$ and $5^{41} - 1$.

Other applications of M. Aurifeuille's formulæ are

(i) when $\dfrac{ab}{5}$ in $\dfrac{a^5 - b^5}{a - b}$ is a square, *i.e.* when $a = 5x^2$, $b = y^2$.

Thus

$$\frac{5^5x^{10} - y^{10}}{5x^2 - y^2} = N_5(y^2 - 5x^2\rho^2) = N_5(y - x\rho\surd 5)\,N_5(y + x\rho\surd 5),$$

but $\surd 5 = \rho + \rho^4 - \rho^2 - \rho^3$;

therefore

$$\frac{5^5x^{10}-y^5}{5x^2-y^2}=N_5\{y-x(1-\rho^2)(1-\rho^4)\}$$

$$\times N_5\{y+x(1-\rho^3)(1-\rho^4)\} \ldots\ldots\ldots(1).$$

(ii) when $\frac{ab}{10}$ in $\frac{a^{10}+b^{10}}{a+b^2}$ is a square, *i.e.* when

(α) $a=10x^2$, $b=y^2$; (β) $a=5x^2$, $b=2y^2$.

Thus $$\frac{10^{10}x^{20}+y^{20}}{10^2x^4+y^4}=N_5(y^4+10^2x^4\rho^4)$$

$$=N_5(y^2-2xy\rho\sqrt{5}+10x^2\rho^2)\,N_5(y^2+2xy\rho\sqrt{5}+10x^2\rho^2)$$

$$=N_5\{y^2-2xy(1-\rho^2)(1-\rho^4)+10x^2\rho^2\}$$

$$\times N_5\{y^2+2xy(1-\rho^3)(1-\rho^4)+10x^2\rho^2\} \ldots\ldots(2).$$

Similarly

$$\frac{5^{10}x^{20}+2^{10}y^{20}}{5^2x^4+2^2y^4}=N_5\{2y^2-2xy(1-\rho^2)(1-\rho^4)+5x^2\rho^2\}$$

$$\times N_5\{2y^2+2xy(1-\rho^3)(1-\rho^4)+5x^2\rho^2\} \ldots(3).$$

In (1), (2), (3), the absolute term is correctly fixed, since in all three cases $\phi(\rho)-\phi(1)\equiv 0$ mod. $(1-\rho)^2$, 5 and 10 being each multiples of $(1-\rho)^2$;

(1) gives 2 a quintic residue of each norm, if x be even,

(2) gives 2 a quintic residue of each norm always,

(3) gives 2 a quintic non-residue always;

thus $x=y=1$ in (2) gives 2 a quintic residue of the factors of $\frac{10^{10}+1}{10^2+1}$, viz. 3541 and 27961.

As an example of the division of one quintic integer by another, one may take the algebraical prime factor of $11^{15}+1$, which is a multiple of 31, since 11 is a primitive root of 31. Thus

$$\frac{(11^{15}+1)(11+1)}{(11^3+1)(11^5+1)}=N_5\left(\frac{11^2+\rho^2}{11+\rho}\right)=N_5(121-11\rho+\rho^2).$$

Hence $121 - 11\rho + \rho^2$ is divisible by one of the four quintic factors of 31, $2-\rho$, $2-\rho^2$, $2-\rho^3$, or $2-\rho^4$

Now $$11^3 + 2 = 1333 = 31.43,$$

therefore $$11^3 + \rho^3 \text{ is divisible by } 2-\rho^3,$$

i. e. $$121 - 11\rho^6 + \rho^{12} \text{ is divisible by } 2-\rho^3,$$

or $$121 - 11\rho^2 + \rho^4 \text{ by } 2-\rho;$$

in fact

$$121 - 11\rho^2 + \rho^4 = 93 + 44(4-\rho^2) - (16-\rho^4)$$

$$=(2-\rho)\{3(2-\rho^2)(2-\rho^3)(2-\rho^4) + 44(2+\rho) - (2+\rho)(4+\rho^2)\},$$

and the norm of the factor in brackets is the quotient by 31 of the algebraical prime factor.

This norm is found to be the prime 7537711, the reciprocal factor being $-1230 + 1890\rho - 24\rho^2 + 292\rho^3 - 927\rho^4$.

25. Jacobi's method, with extensions similar to those explained for the case of quintic integers, makes it possible to form septimic integers, or, in fact, λ^{ic} integers, where λ is any prime greater than 5.

From these reciprocal factors can be formed, after applying Kummer's two conditions for a primary integer, viz.;

$$\phi(\rho) - \phi(1) \equiv 0 \text{ mod. } (1-\rho)^2,$$
$$\phi(\rho)\,\phi(\rho^{-1}) - \phi(1)^2 \equiv 0 \text{ (mod. } \lambda).$$

The reciprocal factor $f(\rho)$, when λ is any uneven prime, is the product $\phi(\rho)\,\phi(\rho^2)\,\phi(\rho^3)\,(\ldots\ldots)\,\phi(\rho^{\frac{1}{2}(\lambda-1)})$ or any product derived from it by substituting ρ^i for ρ.

Pepin's condition that 2 should be a λ^{ic} residue of a prime, is that the absolute term in the reciprocal factor should be the uneven coordinate, all but one being necessarily even. He also gives the rule for 3 to be a 7^{ic} residue of a prime, viz.

$$A_1 \equiv A_2 \equiv A_4 \text{ (mod. 7)},$$
$$A_3 \equiv A_5 \equiv A_6 \text{ (mod. 7)}.$$

Comparing this with the criteria when $\lambda = 5$, they appear to be particular cases of

$$A_1 \equiv A_{g^2} \equiv A_{g^4} \equiv \ldots \text{ (mod. } \lambda),$$
$$A_g \equiv A_{g^3} \equiv A_{g^5} \equiv \ldots \text{ (mod. } \lambda),$$

g being a primitive root of λ.

As an example of these criteria for 3 to be a 7^{ic} residue of a prime, one may take

$$1597 = \frac{1}{7}\,\frac{5^7 + 2^7}{5+2} = N_7\left(\frac{5+2\rho}{1-\rho}\right) = N_7\left(\frac{7}{1-\rho} - 2\right),$$

from which on forming the reciprocal factor the conditions are found to be satisfied.

In a similar way Professor Lloyd Tanner suggests as a deduction from his 5^{ic} criterion for 5, the λ^{ic} criterion for λ, by noting that since $2^5 \equiv 7 \pmod{25}$, 7 is a quintic residue of 25, 2 being a primitive root both of 5 and 25, and the condition that 5 may be a 5^{ic} residue of p becomes

$$a_1 + 2^5 a_2 + 2^{10} a_4 + 2^{15} a \equiv 0 \pmod{5^2}.$$

Hence when $\lambda = 7$, since 3 is a primitive root of 7 and 49, the 7^{ic} criterion for 7 becomes

$$a_1 + 3^7 a_3 + 3^{14} a_9 + 3^{21} a_{27} + 3^{28} a_{81} + 3^{35} a_{243} \equiv 0 \pmod{49};$$

or, since $3^7 \equiv -18 \pmod{49}$ and $3^{14} \equiv -19 \pmod{49}$,

$$(a_1 - a_6) - 18\,(a_3 - a_4) - 19\,(a_2 - a_5) \equiv 0 \pmod{49}.$$

The general criterion appears to be (g being a primitive root of λ and λ^2)

$$a_1 + g^\lambda a_g + g^{2\lambda} a_{g^2} + g^{3\lambda} a_{g^3} + \ldots + g^{(\lambda-2)\lambda} a_{g^{\lambda-2}} \equiv 0 \pmod{\lambda^2};$$

the left-hand member of the congruence being equal to $\phi'(1)$, when $\phi(\rho)$ is written in the form

$$a_0 + a_g \rho^{g^\lambda} + a_{g^2} \rho^{g^{2\lambda}} + a_{g^3} \rho^{g^{3\lambda}} + \ldots\, a_{g^{\lambda-2}} \rho^{g^{(\lambda-2)\lambda}},$$

a form which is equivalent to the ordinary form, since

$$g^\lambda \equiv g,\ g^{2\lambda} \equiv g^2,\ g^{3\lambda} \equiv g^3,\ \ldots,\ g^{(\lambda-2)\lambda} \equiv g^{\lambda-2} \pmod{\lambda},$$

all reduce to $g^{\lambda-1} \equiv 1 \pmod{\lambda}$.

The laws of reciprocity established by Pepin give the λ^{ic} criteria for primes of the form $2\lambda m + 1$ or $2\lambda m - 1$; thus if $\lambda = 7$, the law of septimic reciprocity gives the 7^{ic} criteria for 29 and 13; the result, however, is too complicated to be of much use in numerical calculations.

Pepin does not give any criteria when λ is a power of a prime, *e.g.* 9, 25, 27, ..., so that, as he points out, there is still

something required to complete the investigation; the author, however, has obtained some empirical results for $\lambda = 9$; and Col. Cunningham has prepared tables of factors of $(X^\lambda - Y^\lambda) \div (X - Y)$, when $\lambda = 5, 7, 11$, &c., which supply numerous tests of the rules developed in this paper, applied to the norms of binomial λ^{ic} integers.

CORRIGENDA AND ADDENDA IN FORMER PORTION OF PAPER

(Vol. xxv., pp. 1–44).

p. 4, line 3, *for* $a^n - 1$, *read* $a^{\frac{1}{2}n} + 1$.
line 18, *for* the even, *read* even the.
p. 7, line 2 from foot of text, *for* $(3^{2m+1} + 1)$, *read* $(3^{2m+1} + 1)^2$.
last line of text, *for* 3^m, *read* $3^m + 1$.
p. 9, line 11, *for* $m + 1$, *read* $2m + 1$.
Note at foot, *for* 3.10^{12}, *read* 3.101^2.
p. 10, line 14 from bottom, *for* $-(a^m - 1)^2$, *read* $-a(a^m - 1)^2$.
p. 11, lines 13, 16, &c., *for* $8m \pm 1$, $8m + 3$, $8m + 5$, *read* $8M \pm 1$, $8M + 3$, $8M + 5$.
p. 13, line 2, *for* $r^{\lambda-1}(r - 1)$, *read* $r^{\gamma-1}(r - 1)$.
line 3 from bottom, *for* (mod. 1), *read* (mod. p).
p. 14, line 12, *omit* 12.
line 7 from bottom, *for* 13, *read* 12.
line 13 from bottom, *for* aritmetical, *read* arithmetical.
p. 15, line 5, *for* 2^3, *read* 2^4.
p. 16, line 4 from bottom, *for* 3, *read* 5.
p. 18, line 5, *for* $\left(\frac{2}{p}\right)$, *read* $\left(\frac{2}{p}\right)_4$.
p. 20, line 6, *for* 6673, *read* 9721.
line 7, *for* $2^{273} + 1$, *read* $2^{405} + 1$.
p. 21, line 13, *for* $353 = 17^2 + 8^2$, *read* $313 = 13^2 + 12^2$; *for* (mod. 353), *read* (mod. 313).
lines 14 and 16, *for* 353, *read* 313.
p. 22, line 4 from bottom, *for* $\left(-\frac{3}{p}\right)$, *read* $\left(-\frac{3}{p}\right)_4$.
p. 23, line 3, *for* odd or even, *read* even or odd.
line 8, *for* since $(1 + 8 = 9)$, *read* $= \frac{1}{4}(23^2 + 27.3^2)$; *for* $\left(\frac{2}{193}\right)_{12}$, *read* $\left(\frac{3}{193}\right)_{12}$.
line 10, *for* 3.2^3, *read* 3.2^6.
line 17, *for* 188^2, *read* 108^2.
p. 24, last line, *for* $5^3 + 1$, *read* $5^2 + 1$.
p. 25, line 2 from bottom, *for* is the sum of, *read* depends on.
p. 27, line 10, *for* 6^{3m+3}, *read* 6^{6m+3}.
line 6 from bottom, *for* 7^w, *read* 7^m.
line 5 from bottom, *for* 2, *read* 3.
p. 29, line 15, *for* $5(4\gamma \pm 3)_2$, *read* $5(4\gamma \pm 3)^2$.
p. 30, line 10, *for* 588,2353, *read* 5882353.
line 15, *for* $8n + 1$, *read* $16n + 1$.

p. 31, lines 7, 11, 14, 26, *for* Loof, *read* Looff.
line 24, *omit* $+ (10^2 + 1)$.
line 28, *for* $10^{29} + 1$, *read* $10^{29} - 1$.
line 29, *for* specific factors, presumably prime factors, *read* specific prime factors.
p. 32, line 14, *for* $= 1000, = 2500$, *read* $< 1000, < 2500$.
p. 34, line 3, *for* n, *read* x.
p. 35, line 15 from bottom, *omit* is.
p. 36, line 13, *for* roots, of, *read* roots of.
p. 37, line 6 from bottom, *for* $a^3 - b$, *read* $a^3 - b^3$.
line 5 from bottom, *for* $a^4 - b^4$, *read* $a^4 + b^4$.
p. 38, line 14, *for* 11.7^2, *read* 11.71^2.
p. 42, *omit* lines 12 and 13.
p. 43, col. of $2^n - 1$, line of 47, *for* 2251, *read* 2351.
col. of $3^n - 1$, line of 17, *for* x, *read* 1871.34511.
line of 45, *for* x, *read* 1621.927901.
col. of $5^n - 1$, line of 11, *for* x, *read* 12207031.
line of 16, *for* 11439, *read* 11489.
line of 25, *for* x, *read* 9384251.
line of 31, *for* x, *read* $1861.x$.
line of 32, *for* x, *read* 29423041.
line of 34, *for* x, *read* 41540861
col. of $6^n - 1$, line of 2, *for* 5.7, *read* 7.
line of 11, *for* x, *read* 3154757.
line of 23, *for* x, *read* $3221.x$.
line of 49, *for* $883.x$, *read* x
p. 44, col. of $7^n - 1$, line of 11, *for* x, *read* 1123.293459.
line of 22, *for* x, *read* 10746341.
line of 27, *for* x, *read* 2377.2583253.
line of 30, *for* x, *read* 6568801.
line of 33, *for* x, *read* $3631.x$.
line of 49, *for* x, *read* $3529.x$.
col. of $10^n - 1$, line of 29, *for* x, *read* $62003.x$.
col. of $11^n - 1$, line of 25, *for* x, *read* $5^2.3001.x$.
line of 28, *for* x, *read* $1933.x$.
line of 30, *for* x, *read* 7537711.
col. of $12^n - 1$, line of 16, *for* 26053, *read* 260753.
line of 20, *for* x, *read* $5^2.x$.
line of 44, *for* $3697.x$, *read* $2377.3697.x$.
line of 49, *insert* x.

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THE ELECTRIC AND MAGNETIC IMAGES OF A MULTIPLE POINT IN A SPHERE.

By *E. G. Gallop, M.A.*

CLERK MAXWELL shows in his *Treatise on Electricity and Magnetism*, Chap. IX, how a harmonic potential of negative degree may be regarded as due to a multiple point formed of the assemblage of a number of infinitely near poles. If a sphere at potential zero is placed in a given field of force, the potential of the induced electricity at any external point may be regarded as due to a number of multiple points at the centre of the sphere. If two or more spheres are in the field it is necessary to find the successive images of these multiple points in order to determine completely the distribution of potential.

These multiple points may be classified as zonal, sectorial, or tesseral, according as the potentials due to them involve a zonal, sectorial, or tesseral spherical harmonic.

It is proposed to determine the electric and magnetic images of such multiple points in a sphere.

The following principle will be frequently used. If the potential V of a given electric field involve a parameter h, and the electricity induced on a conductor at zero potential, whose shape is independent of h, produce potential U at any external point, then when the potential of the field is $\frac{\partial V}{\partial h}$ the potential due to the electricity induced on the same conductor will be $\frac{\partial U}{\partial h}$.

§ 1. *Electric image of a zonal point.*

Let O be the centre of a conducting sphere of radius a. Let A be any given external point, and B its image in the sphere. Let $OA = h$ and $OB = c$, so that $ch = a^2$.

Take the axis of z along OA, and the axes of x and y through O perpendicular to OA. Let P be any point (x, y, z); let $AP = r$, $BP = s$, angle $BAP = \theta$, angle $ABP = \phi$, so that

$$r^2 = x^2 + y^2 + (z - h)^2, \quad s^2 = x^2 + y^2 + (z - c)^2.$$

Let P_n, where n is a positive integer, denote the zonal harmonic of order n for the angle θ, and Q_n the same harmonic for the angle ϕ, so that

$$P_n = \frac{1}{n!} r^{n+1} \frac{\partial^n}{\partial z^n}\left(\frac{1}{r}\right) = \frac{(-1)^n}{n!} r^{n+1} \frac{\partial^n}{\partial h^n}\left(\frac{1}{r}\right);$$

$$Q_n = \frac{(-1)^n}{n!} s^{n+1} \frac{\partial^n}{\partial z^n}\left(\frac{1}{s}\right) = \frac{1}{n!} s^{n+1} \frac{\partial^n}{\partial c^n}\left(\frac{1}{s}\right).$$

It will be noticed that the axes of P_n and Q_n are in the negative and positive directions of the axis of z respectively.

Let a point be placed at A which produces potential P_n/r^{n+1}. Such a point will be called a zonal point of unit strength and of order n. It is required to determine the potential U due to the electricity induced on the sphere when at potential zero by this zonal point.

Now if a unit pole, producing potential $1/r$, is placed at A, the potential due to the induced electricity is

$$-\frac{a}{h}\frac{1}{s} = -\frac{c}{a}\frac{1}{s}.$$

Hence, when a zonal point producing potential

$$\frac{P_n}{r^{n+1}} = \frac{(-1)^n}{n!}\frac{\partial^n}{\partial h^n}\frac{1}{r}$$

is placed at A, the potential due to the induced electricity is

$$-\frac{(-1)^n}{n!}\left(\frac{\partial}{\partial h}\right)^n\left(\frac{c}{a}\frac{1}{s}\right) \dots\dots\dots\dots\dots (1).$$

Now, by a formula given in Schlömilch's *Compendium der Höheren Analysis*, II, p. 6, if $xy = 1$,

$$(-1)^n\left(\frac{d}{dx}\right)^n F(y) = y^{2n}\left(\frac{d}{dy}\right)^n F(y) + (n-1)(n)_1 y^{2n-1}\left(\frac{d}{dy}\right)^{n-1} F(y)$$
$$+ (n-1)(n-2)(n)_2 y^{2n-2}\left(\frac{d}{dy}\right)^{n-2} F(y) + \dots,$$

where $(n)_r$ denotes the number of combinations of n things r at a time. Hence, since $ch = a^2$,

$$(-1)^n\left(\frac{\partial}{\partial h}\right)^n F(c) = \frac{1}{a^{2n}}\left[c^{2n}\left(\frac{\partial}{\partial c}\right)^n F(c) + (n-1)(n)_1 c^{2n-1}\left(\frac{\partial}{\partial c}\right)^{n-1} F(c)\right.$$
$$\left. + (n-1)(n-2)(n)_2 c^{2n-2}\left(\frac{\partial}{\partial c}\right)^{n-2} F(c) + \dots\right] \dots (2).$$

Therefore

$$(-1)^n\left(\frac{\partial}{\partial h}\right)^n\left(\frac{c}{a}\,\frac{1}{s}\right)=\frac{1}{a^{2n+1}}\left[c^{2n}\left(\frac{\partial}{\partial c}\right)^n\left(\frac{c}{s}\right)+(n-1)(n)_1c^{2n-1}\left(\frac{\partial}{\partial c}\right)^{n-1}\left(\frac{c}{s}\right)\right.$$

$$\left.+\,(n-1)\,(n-2)\,(n)_2\,c^{2n-2}\left(\frac{\partial}{\partial c}\right)^{n-2}\left(\frac{c}{s}\right)+\dots\right].$$

Now $$\left(\frac{\partial}{\partial c}\right)^p\left(\frac{c}{s}\right)=c\left(\frac{\partial}{\partial c}\right)^p\frac{1}{s}+p\left(\frac{\partial}{\partial c}\right)^{p-1}\frac{1}{s}.$$

The coefficient of $\left(\frac{\partial}{\partial c}\right)^{n-p}\frac{1}{s}$ in the last expression is therefore

$$(n-1)\,(n-2)\dots(n-p)\,(n)_p\,\frac{c^{2n-p+1}}{a^{2n+1}}$$

$$+\,(n-1)\dots(n-p+1)\,(n)_{p-1}\,(n-p+1)\,\frac{c^{2n-p+1}}{a^{2n+1}}$$

$$=\frac{c^{2n-p+1}}{a^{2n+1}}\,\frac{n\,(n-1)^2\,(n-2)^2\dots(n-p+1)^2}{p\,!}\,[(n-p)+p]$$

$$=\frac{n^2\,(n-1)^2\dots(n-p+1)^2}{p\,!}\,\frac{c^{2n-p+1}}{a^{2n+1}}.$$

The potential of the induced electricity is therefore

$$-\frac{1}{n\,!}\,\frac{1}{a^{2n+1}}\left[c^{2n+1}\left(\frac{\partial}{\partial c}\right)^n\frac{1}{s}+\frac{n^2}{1\,!}c^{2n}\left(\frac{\partial}{\partial c}\right)^{n-1}\frac{1}{s}+\frac{n^2\,(n-1)^2}{2\,!}\left(\frac{\partial}{\partial c}\right)^{n-2}\frac{1}{s}+\dots\right]$$

$$=-\left[\frac{a^{2n+1}}{h^{2n+1}}\,\frac{Q_n}{s^{n+1}}+n\,\frac{a^{2n-1}}{h^{2n}}\,\frac{Q_{n-1}}{s^n}+\frac{n\,(n-1)}{2\,!}\,\frac{a^{2n-3}}{h^{2n-1}}\,\frac{Q_{n-2}}{s^{n-1}}+\dots\right].$$

Hence the electric image of the zonal point at A, of order n and strength unity, consists of $n+1$ zonal points placed at the the image of A and of orders from 0 to n, the strength of the point of p^{th} order being

$$-\frac{n\,!}{p\,!\,(n-p)\,!}\,\frac{a^{2p+1}}{h^{n+p+1}}.$$

The result indicates how relatively unimportant are the harmonics of high order when h is large compared with a.

§ 2. *Electric image of a sectorial point.*

Let the potential of a sectorial point at A be

$$V = \frac{(2\sigma)!}{2^\sigma \sigma!} \frac{\sin^\sigma \theta \cos^\sigma \chi}{r^{\sigma+1}} = \frac{\sin^\sigma \theta \cos^\sigma \chi}{r^{\sigma+1}} \frac{d^\sigma P_\sigma}{d\lambda^\sigma},$$

where $\lambda = \cos\theta$ and χ is the angle which the plane APB makes with the plane xz. This is the form of sectorial harmonic given in Ferrers' *Spherical Harmonics*.

Such a point will be called a sectorial point of unit strength. Its potential may also be written

$$V = (-1)^\sigma 2^\sigma D_\sigma \left(\frac{1}{r}\right),$$

where, if $\xi = x + iy$ and $\eta = x - iy$,

$$D_\sigma = \tfrac{1}{2} \left\{ \left(\frac{\partial}{\partial \xi}\right)^\sigma + \left(\frac{\partial}{\partial \eta}\right)^\sigma \right\}.$$

Assume for the potential of the induced electricity

$$U = -C \frac{(2\sigma)!}{2^\sigma \sigma!} \frac{\sin^\sigma \phi \cos \sigma\chi}{s^{\sigma+1}}.$$

Since U satisfies Laplace's equation, this assumption will be correct provided it makes $U + V$ vanish over the surface of the sphere. This will be the case if, for points on the sphere

$$\frac{\sin^\sigma \theta}{\sin^\sigma \phi} \frac{s^{\sigma+1}}{r^{\sigma+1}} \text{ or } \frac{s^{2\sigma+1}}{r^{2\sigma+1}} = C;$$

that is, if $$C = \left(\frac{a}{h}\right)^{2\sigma+1}.$$

The result will be precisely similar if

$$V = \frac{(2\sigma)!}{2^\sigma \sigma!} \frac{\sin^\sigma \theta \sin \sigma\chi}{r^{\sigma+1}}.$$

Hence the electric image of a sectorial point of unit strength at A is a sectorial point of the same type placed at the image of A, its strength being

$$-\left(\frac{a}{h}\right)^{2\sigma+1}.$$

§ 3. *Electric Image of a Tesseral Point.*

Let there be a tesseral point at A producing potential

$$V = \frac{Y_n^{(\sigma)}(\theta, \chi)}{r^{n+1}} = \frac{(-1)^n 2^\sigma}{(n-\sigma)!}\left(-\frac{\partial}{\partial z}\right)^{n-\sigma} D_\sigma\left(\frac{1}{r}\right)$$

$$= \frac{(-1)^n 2^\sigma}{(n-\sigma)!}\left(\frac{\partial}{\partial h}\right)^{n-\sigma} D_\sigma\left(\frac{1}{r}\right),$$

so that, if $\cos\theta = \lambda$,

$$Y_n^{(\sigma)}(\theta, \chi) = \cos\sigma\chi \sin^\sigma\theta \frac{d^\sigma P_n}{d\lambda^\sigma},$$

which is the form of tesseral harmonic adopted in Ferrers' *Spherical Harmonics*. It will be noted that the $(n-\sigma)$ coincident axes of $Y_n^{(\sigma)}$ are in the negative direction of the axis of z.

Such a point will be called a tesseral point of order n, type σ and strength unity.

Now it was proved in § 2 that, when the potential of the field is

$$D_\sigma\left(\frac{1}{r}\right),$$

the potential of the induced electricity is

$$-\left(\frac{c}{a}\right)^{2\sigma+1} D_\sigma\left(\frac{1}{s}\right),$$

Hence the potential of the electricity induced by the tesseral point will be

$$-\frac{(-1)^n 2^\sigma}{(n-\sigma)!}\left(\frac{\partial}{\partial h}\right)^{n-\sigma}\left\{\left(\frac{c}{a}\right)^{2\sigma+1} D_\sigma\left(\frac{1}{s}\right)\right\} \quad\text{............(3).}$$

For the moment write m for $n-\sigma$. Then by (2) of § 1,

$$(-1)^m\left(\frac{\partial}{\partial h}\right)^m\left\{\left(\frac{c}{a}\right)^{2\sigma+1} D_\sigma\left(\frac{1}{s}\right)\right\}$$

$$= \frac{1}{a^{2m+2\sigma+1}}\left[c^{2m}\left(\frac{\partial}{\partial c}\right)^m + (m-1)(m)_1 c^{2m-1}\left(\frac{\partial}{\partial c}\right)^{m-1}\right.$$

$$\left. + (m-1)(m-2)(m)_2 c^{2m-2}\left(\frac{\partial}{\partial c}\right)^{m-2} + \ldots\right] c^{2\sigma+1} D_\sigma\left(\frac{1}{s}\right) \quad\text{...(4).}$$

Expanding

$$\left(\frac{\partial}{\partial c}\right)^m \left\{c^{2\sigma+1} D_\sigma \left(\frac{1}{s}\right)\right\},$$

and similar terms, we shall find for the coefficient of

$$\left(\frac{\partial}{\partial c}\right)^{m-p} D_\sigma \left(\frac{1}{s}\right),$$

$$\frac{c^{2m+2\sigma+1-p}}{a^{2m+2\sigma+1}} \Big[(m)_p (2\sigma+1)\, 2\sigma \ldots (2\sigma+2-p)$$
$$+ (m-1)\,(m)_1\,(m-1)_{p-1}\,(2\sigma+1)\, 2\sigma \ldots (2\sigma+3-p)$$
$$+ (m-1)\,(m-2)\,(m)_2\,(m-2)_{p-2}\,(2\sigma+1)\, 2\sigma \ldots (2\sigma+4-p)$$
$$+ \ldots \text{ to } p+1 \text{ terms}\Big].$$

The expression in square brackets may be written

$$m\,(m-1)\ldots(m-p+1)\left[\frac{(2\sigma+1)\,2\sigma\ldots(2\sigma+2-p)}{p\,!}\right.$$
$$+ \frac{(m-1)}{1\,!}\,\frac{(2\sigma+1)\,2\sigma\ldots(2\sigma+3-p)}{(p-1)\,!}$$
$$\left. + \frac{(m-1)\,(m-2)}{2\,!}\,\frac{(2\sigma+1)\,2\sigma\ldots(2\sigma+4-p)}{(p-2)\,!} + \ldots\right].$$

The last expression in square brackets is equal to the coefficient of x^p in the expansion of

$$(1+x)^{2\sigma+1}\,(1+x)^{m-1} \text{ or } (1+x)^{m+2\sigma},$$

and is therefore equal to

$$\frac{(m+2\sigma)\,!}{p\,!\,(m+2\sigma-p)\,!}.$$

Hence the coefficient of

$$\left(\frac{\partial}{\partial c}\right)^{m-p} D_\sigma \left(\frac{1}{s}\right)$$

in (4) is equal to

$$\frac{c^{2m+2\sigma-p+1}}{a^{2m+2\sigma+1}}\,\frac{m\,!}{p\,!\,(m-p)\,!}\,\frac{(m+2\sigma)\,!}{(m+2\sigma-p)\,!}.$$

Writing p for $m-p$ and replacing m by $n-\sigma$, we find that the expression (3) becomes

$$-\frac{(-1)^\sigma 2^\sigma}{(n-\sigma)!}\Sigma_{p=0}^{p=n-\sigma}\frac{c^{n+\sigma+p+1}}{a^{2n+1}}\frac{(n-\sigma)!}{p!(n-\sigma-p)!}\frac{(n+\sigma)!}{(p+2\sigma)!}\left(\frac{\partial}{\partial c}\right)^p D_\sigma\frac{1}{s}$$

$$=-\Sigma_{p=0}^{p=n-\sigma}\frac{(n+\sigma)!}{(n-\sigma-p)!(p+2\sigma)!}\frac{c^{n+\sigma+p+1}}{a^{2n+1}}\frac{(-1)^{p+\sigma}2^\sigma}{p!}\left(\frac{\partial}{\partial z}\right)^p D_\sigma\left(\frac{1}{s}\right)$$

$$=-\Sigma_{p=0}^{p=n-\sigma}\frac{(n+\sigma)!}{(n-\sigma-p)!(p+2\sigma)!}\frac{c^{n+\sigma+p+1}}{a^{2n+1}}\frac{Y_{p+\sigma}^{(\sigma)}(\phi,\chi)}{r^{p+\sigma+1}}.$$

The electric image of a tesseral point at A, of order n, type σ and unit strength, therefore consists of $n-\sigma+1$ tesseral points placed at the image of A, of orders varying from σ to n, but all of the same type σ, the strength of the point of order $p+\sigma$ being

$$-\frac{(n+\sigma)!}{(n-\sigma-p)!(p+2\sigma)!}\frac{a^{2p+2\sigma+1}}{h^{n+p+\sigma+1}}.$$

It will be noticed that the p coincident axes of $Y_{p+\sigma}^{(\sigma)}(\phi,\chi)$ are opposite in direction to the $(n-\sigma)$ coincident axes of $Y_n^{(\sigma)}(\theta,\chi)$. The particular harmonic $Y_\sigma^{(\sigma)}(\phi,\chi)$ is, of course, a sectorial harmonic.

§ 4. *Magnetic Image of a Sectorial Point.*

Let the sphere described in § 1 be of constant magnetic permeability μ, and let it be placed in a medium of unit permeability. Let a sectorial point at A produce magnetic potential

$$V=\frac{Y^{(\sigma)}(\theta,\chi)}{r^{\sigma+1}}=\frac{(2\sigma)!}{2^\sigma\sigma!}\frac{\sin^\sigma\theta\cos\sigma\chi}{r^{\sigma+1}}=(-1)^\sigma 2^\sigma D_\sigma\left(\frac{1}{r}\right).$$

It is required to determine the magnetic potential Ω at points outside the sphere due to the magnetization induced in the sphere.

Let $OP=R$, and angle $AOP=\Theta$. Then it is easily seen that

$$V=\frac{(2\sigma)!}{2^\sigma\sigma!}\frac{R^\sigma}{r^{2\sigma+1}}\sin^\sigma\Theta\cos\sigma\chi.$$

Now, writing $\cos\Theta=\gamma$, we have

$$\frac{1}{r^{2\sigma+1}}=\frac{1}{(R^2-2Rh\gamma+h^2)^{\frac{1}{2}(2\sigma+1)}}$$

$$=\frac{1}{1.3.5\ldots(2\sigma-1)}\frac{1}{R^\sigma h^\sigma}\left(\frac{d}{d\gamma}\right)^\sigma\frac{1}{(R^2-2Rh\gamma+h^2)^{\frac{1}{2}}}$$

$$=\frac{2^\sigma\sigma!}{(2\sigma)!}\frac{1}{R^\sigma h^{\sigma+1}}\left(\frac{d}{d\gamma}\right)^\sigma\left[1+\frac{R}{h}P_1+\frac{R^2}{h^2}P_2+\ldots\right]$$

$$=\frac{2^\sigma\sigma!}{(2\sigma)!}\frac{1}{R^\sigma h^{\sigma+1}}\left[\left(\frac{R}{h}\right)^\sigma\frac{d^\sigma P_\sigma}{d\gamma^\sigma}+\left(\frac{R}{h}\right)^{\sigma+1}\frac{d^\sigma P_{\sigma+1}}{d\gamma^{\sigma+1}}+\ldots\right],$$

where P_1, P_2, ... denote zonal harmonics of the angle Θ.

Hence $$V=\frac{\sin^\sigma\Theta\cos\sigma\chi}{h^{\sigma+1}}\Sigma_{n=\sigma}^{n=\infty}\left(\frac{R}{h}\right)^n\frac{d^\sigma P_n}{d\gamma^\sigma}.$$

Now when the potential of the external field is ΣAR^nY_n, where Y_n is a spherical harmonic, the external potential due to the induced magnetization is easily seen to be

$$-\frac{\mu-1}{\mu+1}\Sigma\frac{n}{n+\dfrac{1}{\mu+1}}\frac{a^{2n+1}}{R^{n+1}}AY_n.$$

In the present case therefore

$$\Omega=-\frac{\mu-1}{\mu+1}\frac{1}{ah^\sigma}\Sigma_{n=\sigma}^{n=\infty}\frac{n}{n+\dfrac{1}{\mu+1}}\left(\frac{a^2}{hR}\right)^{n+1}\sin^\sigma\Theta\cos\sigma\chi\frac{d^\sigma P_n}{d\gamma^\sigma}$$

$$=-\frac{\mu-1}{\mu+1}\frac{1}{ah^\sigma}\Sigma_{n=\sigma}^{n=\infty}\frac{n}{n+\dfrac{1}{\mu+1}}\left(\frac{c}{R}\right)^{n+1}\sin^\sigma\Theta\cos\sigma\chi\frac{d^\sigma P_n}{d\gamma^\sigma}.$$

To express this result in another form take a variable point Q on OA so that $OQ=\zeta$; and let $QP=\rho$, angle $AQP=\vartheta$, so that $\rho^2=x^2+y^2+(z-\zeta)^2$.

Now $$\frac{\sin^\sigma\vartheta\cos\sigma\chi}{\rho^{\sigma+1}}=\frac{R^\sigma}{\rho^{2\sigma+1}}\sin^\sigma\Theta\cos\sigma\chi,$$

and, as before,

$$\frac{1}{\rho^{2\sigma+1}}=\frac{2^\sigma\sigma!}{(2\sigma)!}\frac{1}{R^\sigma\zeta^\sigma}\left(\frac{d}{d\gamma}\right)^\sigma\frac{1}{(R^2-2R\zeta\gamma+\zeta^2)^{\frac{1}{2}}}$$

$$=\frac{2^\sigma\sigma!}{(2\sigma)!}\frac{1}{R^\sigma\zeta^\sigma}\Sigma_{n=\sigma}^{n=\infty}\frac{\zeta^n}{R^{n+1}}\frac{d^\sigma P_n}{d\gamma^\sigma},$$

since $\zeta<R$.

Therefore

$$\frac{(2\sigma)!}{2^\sigma\sigma!}\zeta^\sigma\frac{\sin^\sigma\vartheta\cos\sigma\chi}{\rho^{\sigma+1}}=\Sigma_{n=\sigma}^{n=\infty}\frac{\zeta^n}{R^{n+1}}\frac{d^\sigma P_n}{d\gamma^\sigma}\sin^\sigma\Theta\cos\sigma\chi,$$

or
$$(-1)^\sigma 2^\sigma\zeta^\sigma D_\sigma\left(\frac{1}{\rho}\right)=\Sigma_{n=\sigma}^{n=\infty}\frac{\zeta^n}{R^{n+1}}\frac{d^\sigma P_n}{d\gamma^\sigma}\sin^\sigma\Theta\cos\sigma\chi.$$

Therefore

$$(-1)^\sigma 2^\sigma\int_0^c\zeta^{\frac{1}{\mu+1}}\frac{\partial}{\partial\zeta}\left\{\zeta^\sigma D_\sigma\left(\frac{1}{\rho}\right)\right\}d\zeta$$

$$=\Sigma_{n=\sigma}^{n=\infty}\frac{1}{n+\dfrac{1}{\mu+1}}\frac{c^{n+\frac{1}{\mu+1}}}{R^{n+1}}\frac{d^\sigma P_n}{d\gamma^\sigma}\sin^\sigma\Theta\cos\sigma\chi.$$

Therefore

$$\Omega=-\frac{\mu-1}{\mu+1}(-1)^\sigma 2^\sigma\frac{c^{\frac{\mu}{\mu+1}}}{ah^\sigma}\int_0^c\zeta^{\frac{1}{\mu+1}}\frac{\partial}{\partial\zeta}\left\{\zeta^\sigma D_\sigma\left(\frac{1}{\rho}\right)\right\}d\zeta\quad\ldots\ldots(5)$$

$$=-\frac{\mu-1}{\mu+1}2^\sigma\frac{c^{\frac{\mu}{\mu+1}}}{ah^\sigma}\int_0^c\zeta^{\frac{1}{\mu+1}}\left[(-1)^\sigma\sigma\zeta^{\sigma-1}D_\sigma\left(\frac{1}{\rho}\right)\right.$$
$$\left.+(-1)^{\sigma+1}\zeta^\sigma\frac{\partial}{\partial z}D_\sigma\left(\frac{1}{\rho}\right)\right]d\zeta$$

$$=-\frac{\mu-1}{\mu+1}\frac{c^{\frac{\mu}{\mu+1}}}{ah^\sigma}\int_0^c\zeta^{\frac{1}{\mu+1}}\left[\frac{\sigma\zeta^{\sigma-1}Y^{(\sigma)}(\vartheta,\chi)}{\rho^{\sigma+1}}+\frac{\zeta^\sigma Y_{\sigma+1}^{(\sigma)}(\vartheta,\chi)}{\rho^{\sigma+2}}\right]d\zeta.$$

The magnetic image of the sectorial point at A may therefore be described as a line distribution of sectorial points of the same type σ, and another line distribution of tesseral points of order $\sigma+1$ and type σ, the line extending from the centre of the sphere to the image of A, and the line-densities at distance ζ from O being respectively

$$-\frac{\mu-1}{\mu+1}\frac{a^{\frac{\mu-1}{\mu+1}}}{h^{\sigma+\frac{\mu}{\mu+1}}}\sigma\zeta^{\sigma-\frac{\mu}{\mu+1}}\text{ and }-\frac{\mu-1}{\mu+1}\frac{a^{\frac{\mu-1}{\mu+1}}}{h^{\sigma+\frac{\mu}{\mu+1}}}\zeta^{\sigma+\frac{1}{\mu+1}}.$$

Or we may integrate (5) by parts and thus obtain the image in a simpler form as a sectorial point at the image of A, of type σ and strength

$$-\frac{\mu-1}{\mu+1}\frac{c^{\sigma+1}}{ah^{\sigma}}=-\frac{\mu-1}{\mu+1}\left(\frac{a}{h}\right)^{2\sigma+1},$$

together with a line distribution of sectorial points, type σ, extending from the centre O to the image of A and of line density at distance ζ from O equal to

$$\frac{\mu-1}{(\mu+1)^2}\frac{a^{\frac{\mu-1}{\mu+1}}}{h^{\sigma+\frac{\mu}{\mu+1}}}\zeta^{\sigma-\frac{\mu}{\mu+1}}.$$

When μ is large this line density is small. Putting $\mu=\infty$, we get the result of § 2.

§ 5. *Magnetic Image of a Tesseral Point.*

Let a tesseral point at A produce potential

$$V=\frac{Y_n^{(\sigma)}(\theta,\chi)}{r^{n+1}}=\frac{\sin^{\sigma}\theta\cos\sigma\chi}{r^{n+1}}\frac{d^{\sigma}P_n}{d\lambda^{\sigma}},$$

$$=\frac{(-1)^n 2^{\sigma}}{(n-\sigma)!}\left(-\frac{\partial}{\partial z}\right)^{n-\sigma}D_{\sigma}\frac{1}{r}$$

$$=\frac{(-1)^n 2^{\sigma}}{(n-\sigma)!}\left(\frac{\partial}{\partial h}\right)^{n-\sigma}D_{\sigma}\left(\frac{1}{r}\right).$$

Here P_n is a zonal harmonic of the angle θ, and $\lambda=\cos\theta$. It is required to determine the external potential Ω of the induced magnetization.

In § 4 it was shown that with a sectorial point at A producing potential

$$D_{\sigma}\left(\frac{1}{\rho}\right)$$

the potential due to the induced magnetization of the sphere is

$$-\frac{\mu-1}{\mu+1}\frac{c^{\frac{1}{\mu+1}}}{ah^{\sigma}}\int_0^c \zeta^{\frac{1}{\mu+1}}\frac{\partial}{\partial\zeta}\left\{\zeta^{\sigma}D_{\sigma}\left(\frac{1}{\rho}\right)\right\}d\zeta$$

$$=-\frac{\mu-1}{\mu+1}\left(\frac{c}{a}\right)^{2\sigma+1}\int_0^1 \eta^{\frac{1}{\mu+1}}\frac{\partial}{\partial\eta}\left\{\eta^{\sigma}D_{\sigma}\left(\frac{1}{\rho}\right)\right\}d\eta,$$

where $\zeta=c\eta$, and therefore $\rho^2=x^2+y^2+(z-c\eta)^2$.

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To obtain Ω we must operate on this expression with

$$\frac{(-1)^n 2^\sigma}{(n-\sigma)!}\left(\frac{\partial}{\partial h}\right)^{n-\sigma}.$$

A precisely similar operation has been carried out in § 3. Applying the result obtained there, we have in the present case

$$\Omega = -(-1)^\sigma 2^\sigma \frac{\mu-1}{\mu+1} \Sigma_{p=0}^{p=n-\sigma} \frac{(n+\sigma)!}{p!\,(n-\sigma-p)!\,(p+2\sigma)!}$$

$$\times \frac{c^{n+\sigma+p+1}}{a^{2n+1}} \int_0^1 \frac{1}{\eta^{\mu+1}} \frac{\partial}{\partial \eta} \left\{\eta^\sigma \left(\frac{\partial}{\partial c}\right)^p D_\sigma \left(\frac{1}{\rho}\right)\right\} d\eta.$$

Now $\qquad \left(\frac{\partial}{\partial c}\right)^p \frac{1}{\rho} = \eta^p \left(\frac{\partial}{\partial \zeta}\right)^p \frac{1}{\rho} = (-1)^p \eta^p \left(\frac{\partial}{\partial z}\right)^p \frac{1}{\rho}.$

Therefore

$$(-1)^\sigma \frac{\partial}{\partial \eta}\left\{\eta^\sigma \left(\frac{\partial}{\partial c}\right)^p D_\sigma\left(\frac{1}{\rho}\right)\right\} = (-1)^{p+\sigma}\frac{\partial}{\partial \eta}\left\{\eta^{p+\sigma}\left(\frac{\partial}{\partial z}\right)^p D_\sigma\left(\frac{1}{\rho}\right)\right\}$$

$$= (-1)^{p+\sigma}(p+\sigma)\,\eta^{p+\sigma-1}\left(\frac{\partial}{\partial z}\right)^p D_\sigma\left(\frac{1}{\rho}\right)$$

$$+ (-1)^{p+\sigma+1} c\eta^{p+\sigma}\left(\frac{\partial}{\partial z}\right)^{p+1} D_\sigma\left(\frac{1}{\rho}\right).$$

Hence

$$\Omega = -\frac{\mu-1}{\mu+1}\Sigma_{p=0}^{p=n-\sigma+1} \frac{(n+\sigma)!\,\{(n+1)(p+\sigma)-\sigma^2\}}{(n-\sigma-p+1)!\,(p+2\sigma)!}$$

$$\times \frac{c^{n+\sigma+p+1}}{a^{2n+1}} \int_0^1 \frac{(-1)^{p+\sigma}\,2^\sigma}{p!}\,\eta^{p+\sigma-\frac{\mu}{\mu+1}} \left(\frac{\partial}{\partial z}\right)^p D_\sigma\left(\frac{1}{\rho}\right) d\eta.$$

For the *numerical* coefficient involved in the type term is

$$\frac{(n+\sigma)!}{p!\,(n-\sigma-p)!\,(p+2\sigma)!}(p+\sigma)$$

$$+ \frac{(n+\sigma)!}{(p-1)!\,(n-\sigma-p+1)!\,(p-1+2\sigma)!}$$

$$= \frac{(n+\sigma)!}{p!(n-\sigma-p+1)!(p+2\sigma)!}\{(p+\sigma)(n-\sigma-p+1)+p(p+2\sigma)\}$$

$$= \frac{(n+\sigma)!}{p!\,(n-\sigma-p+1)!\,(p+2\sigma)!}\{(n+1)(p+\sigma)-\sigma^2\}.$$

Replacing $\zeta\,(=c\eta)$, we now find for Ω

$$\Omega = -\frac{\mu-1}{\mu+1}\sum_{p=0}^{p=n-\sigma+1}\frac{(n+\sigma)!\,\{(n+1)(p+\sigma)-\sigma^2\}}{(n-\sigma-p+1)!\,(p+2\sigma)!}$$

$$\times\frac{c^{n+\frac{\mu}{\mu+1}}}{a^{2n+1}}\int_0^c \zeta^{p+\sigma-\frac{\mu}{\mu+1}}\frac{Y_{p+\sigma}^{(\sigma)}(\vartheta,\chi)}{\rho^{p+\sigma+1}}\,d\zeta$$

$$= -\frac{\mu-1}{\mu+1}\,\frac{a^{\frac{\mu-1}{\mu+1}}}{h^{n+\frac{\mu}{\mu+1}}}\int_0^{\frac{a^2}{h}}\sum_{p=0}^{p=n-\sigma+1}\frac{(n+\sigma)!\,\{(n+1)(p+\sigma)-\sigma^2\}}{(n-\sigma-p+1)!\,(p+2\sigma)!}$$

$$\times\,\zeta^{p+\sigma-\frac{\mu}{\mu+1}}\frac{Y_{p+\sigma}^{(\sigma)}(\vartheta,\chi)}{\rho^{p+\sigma+1}}\,d\zeta \ldots (6).$$

Hence the magnetic image of the tesseral point at A, of order n, type σ, and strength unity, consists of $(n-\sigma+2)$ lines of tesseral points extending from the centre of the sphere to the image of A. These points are all of type σ, but of orders varying from σ to n, the line density at distance ζ from 0, for the points of order $p+\sigma$ being

$$-\frac{\mu-1}{\mu+1}\,\frac{a^{\frac{\mu-1}{\mu+1}}}{h^{n+\frac{\mu}{\mu+1}}}\,\frac{(n+\sigma)!\,\{(n+1)(p+\sigma)-\sigma^2\}}{(n-\sigma-p+1)!\,(p+2\sigma)!}\,\zeta^{p+\sigma-\frac{\mu}{\mu+1}}.$$

In the case when $\sigma=1$, this result agrees with that obtained by Mr. Herman, *On the motion of two spheres in fluid and allied problems*, *Quar. Jour. Math.*, Vol. XXII, 1887, Section V, § 9. But it is not consistent with a more general result stated without proof in the following § 10 of the same paper.

As in § 4 a different form may be given to the image. The integral in the expression (6) may be written

$$\frac{(-1)^\sigma 2^\sigma}{p!}\int_0^c \zeta^{p+\sigma-\frac{\mu}{\mu+1}}\left(\frac{\partial}{\partial\zeta}\right)^p D_\sigma\left(\frac{1}{r}\right)d\zeta.$$

This may be integrated by parts p times, and the image can therefore be described as consisting of $n-\sigma+1$ tesseral points at B the image of A, of type σ and orders varying from σ to n, together with a line distribution of sectorial points, type σ, extending from the centre to the image of A.

When μ is large the line distribution is of small importance, and the strengths of the sectorial points at B approximate to the values found in § 3 for the electric images.

§ 6. *Magnetic image of a zonal point.*

Let a zonal point at A produce magnetic potential

$$\frac{P_n}{r^{n+1}}.$$

The induced magnetization will produce a potential at points outside the sphere whose value may be obtained by putting $\sigma = 0$ in the result of the last section. We thus find

$$\Omega = -\frac{\mu-1}{\mu+1}\,\frac{a^{\frac{\mu-1}{\mu+1}}}{h^{n+\frac{\mu}{\mu+1}}}\int_0^{\frac{a^2}{h}} \zeta^{\frac{1}{\mu+1}}\,\Sigma_{p=0}^{p=n+1}\,\frac{(n+1)\,p\,.\,n!}{(n-p+1)!\,p!}\,\zeta^{p-1}\,\frac{P_p(\vartheta)}{\rho^{p+1}}\,d\zeta$$

$$= -(n+1)\,\frac{\mu-1}{\mu+1}\,\frac{a^{\frac{\mu-1}{\mu+1}}}{h^{n+\frac{\mu}{\mu+1}}}$$

$$\times\int_0^{\frac{a^2}{h}} \zeta^{\frac{1}{\mu+1}}\,\Sigma_{p=0}^{p=n}\,\frac{n!}{p!\,(n-p)!}\,\zeta^p\,\frac{P_{p+1}(\vartheta)}{\rho^{p+2}}\,d\zeta \ldots (7).$$

The image of the zonal point at A therefore consists of $n+1$ line distributions of zonal points extending from the centre of the sphere to the image of A, the orders varying from 1 to $n+1$, and the line density at distance ζ for the $(p+1)^{\text{th}}$ order being

$$-(n+1)\,\frac{\mu-1}{\mu+1}\,\frac{a^{\frac{\mu-1}{\mu+1}}}{h^{n+\frac{\mu}{\mu+1}}}\,\frac{n!}{p!\,(n-p)!}\,\zeta^{p+\frac{1}{\mu+1}}.$$

This result agrees with that given by Mr. Herman in the paper previously cited, §§ 5 and 3, when an obvious misprint is corrected in the final statement.

As in §§ 4, 5 the expression (7) may be interpreted otherwise.

Write

$$\frac{P_{p+1}(\vartheta)}{\rho^{p+2}} = \frac{1}{(p+1)!}\,\frac{\partial^{p+1}}{\partial\zeta^{p+1}}\left(\frac{1}{\rho}\right).$$

Integrate the corresponding term of the integral by parts $p+1$ times, and we obtain for the image $n+1$ zonal points at the image of A, of orders from 0 to n, the strength of the point of q^{th} order being

$$-(-1)^q q!\,\frac{\mu-1}{\mu+1}\,\frac{a^{\frac{\mu-1}{\mu+1}}}{h^{n+\frac{\mu}{\mu+1}}}\,\zeta^{q+\frac{1}{\mu+1}}\,\Sigma_{p=q}^{p=n}(-1)^p\frac{(n+1)!}{(p+1)!\,p!\,(n-p)!}$$

$$\times\left(p+\frac{1}{\mu+1}\right)\left(p-1+\frac{1}{\mu+1}\right)\ldots\left(q+1+\frac{1}{\mu+1}\right);$$

together with a line distribution of magnetic matter extending from the centre to the image of A, the line density at distance ζ from 0 being

$$-(n+1)\frac{\mu-1}{\mu+1}\,\frac{a^{\frac{\mu-1}{\mu+1}}}{h^{n+\frac{\mu}{\mu+1}}}\,\zeta^{-\frac{\mu}{\mu+1}}\,\Sigma_{p=0}^{p=n}(-1)^{p+1}\frac{n!}{p!\,(n-p)!}$$

$$\times\frac{(p+k)(p+k-1)\ldots k}{(p+1)!},$$

where $k=\dfrac{1}{\mu+1}$; and this last expression reduces to

$$\frac{\mu-1}{(\mu+1)^2}\,\frac{a^{\frac{\mu-1}{\mu+1}}}{h^{n+\frac{\mu}{\mu+1}}}\,\frac{1}{n!}$$

$$\times\left(n-\frac{1}{\mu+1}\right)\left(n-1-\frac{1}{\mu+1}\right)\ldots\left(1-\frac{1}{\mu+1}\right)\zeta^{-\frac{\mu}{\mu+1}},$$

since the series included in Σ is the coefficient of x^{n+1} in the expansion of $(1+x)^n\times(1+x)^{-k}$.

When μ is infinite the line distribution disappears, and the strengths of the $n+1$ zonal points have the values found in § 1 for the electric images.

A PROJECTIVE PROOF OF THE ANHARMONIC PROPERTY OF TANGENTS TO A PLANE CUBIC.

By *A. C. Dixon, Sc.D.*

Any plane cubic is the projection of the curve of intersection of two conicoids from some point upon it.

For if the cubic equation is $u_3 + u_2 + u_1 + u_0 = 0$ and the linear factors of u_3 are α, β, γ, we may take the equations to the conicoids as

$$z = \alpha\beta, \quad z\gamma + u_2 + u_1 + u_0 = 0.$$

Let A be the vertex of projection, B any other point of the twisted quartic, b its projection on the plane of the cubic.

Then the points of contact of tangents from b to the cubic are the projections of the points of contact of tangent planes through the chord AB to the quartic.

The theorem to be proved is therefore that the cross-ratio of these tangent planes is constant.

Through AB and the quartic can be described a conicoid. The tangent planes through AB contain generators of this conicoid of the other system which touch the quartic. We have to prove that the cross ratio of those generators of either system of any conicoid through a given quartic, which touch the quartic, is constant.

Let $U = 0$ be one of the four quadric cones through the quartic, and let S be its vertex. Let $V = 0$ be any other of the conicoids.

Consider the projection of the whole from S upon the common polar plane of S with respect to all the conicoids.

The generators of V are projected into the tangents to the conic v which is the trace of V upon this plane. The conic v passes through four fixed points, namely, those in which the given quartic cuts the plane.

The quartic is projected into u, the trace of the cone $U = 0$. Hence the generators of V which touch the quartic are projected into the four common tangents to the conics u and v. The anharmonic ratio of these generators, being that of the points in which they meet a generator of the opposite system, is the same as that of the points where any fifth tangent to v meets these four common tangents.

It is therefore to be proved that, if u is a fixed conic and v a variable one passing through four fixed points on u, the cross ratio of the four points in which any tangent to v meets the four common tangents to u and v is constant. By taking v very near to u we see that this constant must be the cross ratio of the pencil subtended by the four fixed points at any point of u.

The final form to which the theorem is reduced is then the following: the cross ratio of the four common points of two conics, taken on the one, is equal to that of their four common tangents, taken on the other.

This follows at once from the known fact that a third conic can be found with respect to which the two are reciprocals.

ON A POINT IN THE CALCULUS OF VARIATIONS.

By *A. C. Dixon, Sc.D.*

It was pointed out by Bertrand (Liouville's *Journal*, Vol. VII., pp. 55–58, or Todhunter's *History*, p. 346), that the usual justification for the use of an arbitrary multiplier in problems of the isoperimetrical class is unsatisfactory. The following proof appears to be simpler than the one given by Bertrand.

Suppose that $\int_{x_0}^{x_1} Vdx$ is to have a maximum or minimum value under the condition that $\int_{x_0}^{x_1} Udx$ is equal to a given quantity c. Let n be the order of the highest differential coefficients of y, the unknown function, involved in U or V.

The ordinary reduction gives the following forms:

$$\delta \int_{x_0}^{x_1} Vdx = [C]_{x_0}^{x_1} + \int_{x_0}^{x_1} v\delta y\, dx, \text{ and}$$

$$\delta \int_{x_0}^{x_1} Udx = [B]_{x_0}^{x_1} + \int_{x_0}^{x_1} u\delta y\, dx,$$

The forms of u and v are known when that of y is known. Suppose that for any particular form of y under consideration the value of

$$\int_{x_0}^{x_1} uv\,(x_1-x)^n\,(x-x_0)^n\,dx \div \int_{x_0}^{x_1} u^2\,(x_1-x)^n\,(x-x_0)^n\,dx$$

is k. Then k is a constant.

Also $\epsilon\,(v-ku)\,(x_1-x)^n\,(x-x_0)^n$ is a possible form for δy, ϵ being an infinitesimal constant. For this value of δy makes $\delta \int_{x_0}^{x_1} Udx$ vanish.

Accordingly $\delta \int_{x_0}^{x_1} Vdx$ must also vanish for this value of δy, that is, we must have

$$\int_{x_0}^{x_1} v\,(v-ku)\,(x_1-x)^n\,(x-x_0)^n\,dx = 0,$$

since C, like B, vanishes at each limit.

From this equation and the former one

$$\int_{x_0}^{x_1} u\,(v-ku)\,(x_1-x)^n\,(x-x_0)^n\,dx = 0,$$

it follows at once that

$$\int_{x_0}^{x_1} (v-ku)^2\,(x_1-x)^n\,(x-x_0)^n\,dx = 0.$$

In this the subject of integration is always positive unless $v-ku=0$ constantly between the limits.

This equation must therefore be satisfied by y if the requirements of the problem are to be fulfilled.

It is clear also that if y satisfies this equation the ratio

$$\int_{x_0}^{x_1} uv\,(x_1-x)^n\,(x-x_0)^n\,dx \div \int_{x_0}^{x_1} u^2\,(x_1-x)^n\,(x-x_0)^n\,dx$$

will be actually equal to k, so that the value of k is at our disposal.

There is apparently a difficulty if y always satisfies the equation $u=0$ between the limits, as in that case the denominator of k in its original form vanishes. But then

$$\epsilon v\,(x_1-x)^n\,(x-x_0)^n$$

is a possible form for δy, and thus we must have $v=0$, so that the equation

$$v - ku = 0$$

is still satisfied.

One advantage in this proof is the continuity of form of $\delta y/\epsilon$; there is thus, in general, no difficulty in the integrations by parts. If through any discontinuity in the form of y or other cause v or u should be discontinuous, it will often be possible to avoid the difficulty by using a variation of the form

$$\epsilon (v - ku)(x_1 - x)^n (x - x_0)^n (x - \xi)^{2m},$$

ξ being the value of x at which the discontinuity occurs, and m a suitably chosen positive whole number. It will then appear that $v = ku$ except when $x = \xi$.

The proof can also be extended to the case when there are more conditions than one. Suppose that a third integral $\int_{x_0}^{x_1} W dx$ is also to have a constant value, and let the reduced form of $\delta \int_{x_0}^{x_1} W dx$ be $[D]_{x_0}^{x_1} + \int_{x_0}^{x_1} w \delta y \, dx$.

Then by taking the form

$$\epsilon (v - ku - lw)(x_1 - x)^n (x - x_0)^n$$

for δy, we may find as before that

$$v - ku - lw = 0,$$

the values of k and l being originally given by the equations

$$k \int_{x_0}^{x_1} u^2 (x_1 - x)^n (x - x_0)^n dx$$
$$+ l \int_{x_0}^{x_1} uw (x_1 - x)^n (x - x_0)^n dx = \int_{x_0}^{x_1} uv (x_1 - x)^n (x - x_0)^n dx,$$

$$k \int_{x_0}^{x_1} uw (x_1 - x)^n (x - x_0)^n dx$$
$$+ l \int_{x_0}^{x_1} w^2 (x_1 - x)^n (x - x_0)^n dx = \int_{x_0}^{x_1} vw (x_1 - x)^n (x - x_0)^n dx.$$

CONDITIONS THAT A QUADRIC MAY BE ONE-SIGNED.

By Professor *E. J. Nanson.*

In a former paper the conditions that a quadric may be one-signed for all values of the variables which satisfy given linear homogeneous equations were deduced from the well-known conditions of Dr. Williamson for the case in which all variables are independent.

The object of this paper is to give a direct investigation of the general problem.

§ 1. Let $u, = \Sigma_{pq} a_{pq} x_p x_q$, be a quadric in l variables $x_1, \ldots, x_l$ which satisfy the m equations

$$\Sigma_p b_{pr} x_p = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where r has all values from 1 to m. The sign of u depends only on the ratios $x_1 : x_2 : \ldots : x_l$. In determining that sign we may therefore suppose that the x's are subject to the relation

$$\Sigma_p x_p^2 = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Now, subject to the conditions (1), (2), u has $l-m, = n$ say, stationary values which are the roots of the equation in λ,

$$\begin{vmatrix} a_{11}-\lambda, & \ldots, & a_{1l}, & b_{11}, & \ldots, & b_{1m} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{l1}, & \ldots, & a_{ll}-\lambda, & b_{l1}, & \ldots, & b_{lm} \\ b_{11}, & \ldots, & b_{l1}, & 0, & \ldots, & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ b_{1m}, & \ldots, & b_{lm}, & 0, & \ldots, & 0 \end{vmatrix} = 0 \ldots.(3),$$

and the least (greatest) of these stationary values is the minimum (maximum) value of u. Hence it is clear that necessary and sufficient conditions that u may be positive (negative) for all finite values of $x_1, \ldots, x_l$ which satisfy (1) are that all the roots of (3) are positive (negative).

§ 2. Now let Δ_0 denote the determinant (3) and let $\Delta_1, \Delta_2, \Delta_3, \ldots, \Delta_n$ denote the determinants obtained by striking out from the first l rows and columns of Δ_0 any conjugate row and column, a second conjugate row and column, a third conjugate row and column, and so on. We thus obtain a series of functions of λ whose orders diminish

regularly from n to 0. The last of these functions Δ_n is equal to $(-1)^m B_n^2$, where B_n is the determinant formed with those of the coefficients b in the border of Δ_0 which are not struck out in the process of arriving at Δ_n. Now the equations (1) being assumed to be independent the determinants

$$\| b_{pq} \| \quad \begin{pmatrix} p = 1, \ldots, l \\ q = 1, \ldots, m \end{pmatrix}$$

cannot all be zero. It is therefore clear that by suitably choosing the rows and columns to be struck out we can ensure that Δ_n does not vanish.

§ 3. Let I_{pq} denote any minor of Δ_{r-1}, then we know that

$$\Delta_{r-1} \Delta_{r+1} = I_{11} I_{22} - I_{12}^2,$$

and $I_{11} = \Delta_r$. Thus when Δ_r vanishes Δ_{r-1}, Δ_{r+1} have opposite signs. Hence, remembering that Δ_n does not contain λ and is not zero, we see that as λ increases no change of sign can be lost or gained in the series $\Delta_0, \Delta_1, \Delta_2, \ldots, \Delta_n$, except when λ passes through a root of Δ_0.

§ 4. By expanding Δ_r as a quadratic function of the minors formed from the border of Δ_r by a double application of Laplace's theorem, we readily see that

$$\Delta_r = \Delta_r' + \text{terms in } \lambda, \lambda^2, \ldots, \lambda^{n-r-1} + (-1)^{l-r} \lambda^{n-r} \Sigma B_r^2$$

where Δ_r' is the value of Δ_r when $\lambda = 0$ and B_r stands for any one of the determinants $\| b_{pq} \|$ of order m which can be formed out of the border of Δ_r. Thus the coefficient of the highest power of Δ in Δ_r is positive or negative according as $l - r$ is even or odd. The coefficients of the highest power of λ in $\Delta_0, \Delta_1, \ldots, \Delta_n$ are therefore alternately positive and negative or alternately negative and positive. Hence as λ passes from $-\infty$ to $+\infty$ the functions $\Delta_0, \Delta_1, \ldots, \Delta_n$ gain n changes of sign.

§ 5 From sections 3, 4 it immediately follows that Δ_0 has n real roots and that, as λ increases, a change of sign once gained in the series $\Delta_0, \Delta_1, \ldots, \Delta_n$ cannot be lost. Hence we see that if k changes of sign are gained by $\Delta_0, \Delta_1, \ldots, \Delta_n$ as λ increases from α to β, then there are exactly k roots of Δ_0 between α and β.

§ 6. We can now find the conditions that the roots of (3) are all positive. In order that these roots may be positive it

is clear that the n changes of sign gained by $\Delta_0, \Delta_1, \dots, \Delta_n$ as λ passes from $-\infty$ to $+\infty$ must all be gained in the interval from 0 to $+\infty$. Thus $\Delta_0', \Delta_1', \dots, \Delta_n'$, must be all positive or all negative. Now $\Delta_n' = \Delta_n = (-1)^m B_n^2$. Hence the required conditions are $(-1)^m\Delta_0', (-1)^m\Delta_1', \dots, (-1)^m\Delta_{n-1}'$ all positive. These then are the conditions that the quadric u may be positive for all values of $x_1, \dots, x_l$ which satisfy (1) and are not all zero.

§ 7. In like manner it is found that the conditions that u is negative are $(-1)^l\Delta_0', (-1)^{l-1}\Delta_1', \dots, (-1)^{l-n+1}\Delta_{n-1}'$ all positive. These conditions also follow from the last section by changing the signs of the coefficients a_{pq}.

§ 8. It will be observed that Δ_0' is the discriminant of the quadric u bordered with the coefficients of the linear equations (1) and that Δ_r' is the determinant obtained by striking out r conjugate rows and columns in Δ_0'. We may also find Δ_r' by making r of the variables zero in u and in (1) and then forming the bordered discriminant from the system reduced.

Again, if we write the linear equation (1) in the form $L_r = 0$, it will be seen that Δ_0' is the discriminant of $u + \Sigma_s \lambda_s L_s$ with respect to the x's and λ's, and that Δ_r' is the discriminant of what $u + \Sigma_s \lambda_s L_s$ becomes when r of the x's are made zero.

Melbourne,
March **20, 1896.**

THE PERIOD-EQUATION OF A CONSTRAINED SYSTEM OSCILLATING ABOUT A POSITION OF EQUILIBRIUM.

By Professor *E. J. Nanson.*

§ 1. THE equation in question may be written

$$\begin{vmatrix} a_{11}\lambda + b_1, & \dots, & a_{1l}\lambda + b_{1l}, & c_{11}, & \dots, & c_{1m} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{l1}\lambda + b_{l1}, & \dots, & a_{ll}\lambda + b_{ll}, & c_{l1}, & \dots, & c_{lm} \\ c_{11}, & \dots, & c_{l1}, & 0, & \dots, & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ c_{1m}, & \dots, & c_{lm}, & 0, & \dots, & 0 \end{vmatrix} = 0 \dots (1),$$

where the coefficients a_{ij} are such that $\Sigma a_{ij}x_i x_j$ is a one-signed positive function and $a_{ij} = a_{ji}$, $b_{ij} = b_{ji}$, $l > m$.

The object of this paper is to shew that the roots of (1) are all real, to find the number of roots within given limits, to deduce the conditions that all the roots are negative, and finally to shew that the results obtained hold when $\Sigma a_{ij}x_i x_j$ is not one-signed, and positive for all values of the x's, but only for such as satisfy the m linear equations

$$\Sigma_i c_{ik} x_i = 0 \quad \text{........................(2).}$$

§ 2. Let Δ_0 denote the determinant in (1), and let Δ_1, Δ_2, Δ_3, &c. denote the determinants obtained by striking out from the first l rows and columns of Δ_0 any conjugate row and column, a second conjugate row and column, a third and so on. Let $l - m = n$, then

$$\Delta_0,\ \Delta_1,\ \Delta_2,\ \ldots,\ \Delta_n \quad \text{..................(3)}$$

are a series of functions of λ whose orders diminish regularly from n to 0, and, by a well-known property of symmetrical determinants, when Δ_r vanishes Δ_{r-1}, Δ_{r+1} have opposite signs.

§ 3. It is clear that $\Delta_n = (-1)^m C^2$, where C is the determinant formed with those elements of the border of Δ_0 which are not struck out in the process of arriving at Δ_n. Now it may be assumed that the determinants $\| c_{ik} \|$ of the m^{th} order which can be formed out of the border of Δ_0 are not all zero, for if they are all zero, then the determinant Δ_0 vanishes identically. It is therefore clear that by suitably choosing the rows and columns to be struck out we can ensure that Δ_n does not vanish.

§ 4. Let $\overline{\Delta}_r$ denote the value of Δ_r when λ is replaced by unity, and each b by zero, so that $\overline{\Delta}_r$ is the coefficient of the highest power of λ in Δ_r. Now suppose the coefficients a_{ij} are such that $\Sigma a_{ij}x_i x_j$ is one-signed and positive for all values of the x's which satisfy (2). Then it is known (see preceding paper) that $(-1)^m \overline{\Delta}_r$ is positive for all values of r from 0 to $n - 1$, and from the last section it follows that $(-1)^m \overline{\Delta}_n$ is positive. Thus the coefficients of the highest power of λ in the functions (3) are all of the same sign. Hence, as λ increases from $-\infty$ to $+\infty$, the functions (3) lose n changes of sign.

§ 5. Since Δ_n cannot vanish, and Δ_{r-1}, Δ_{r+1} have opposite signs when Δ_r vanishes, it follows that the functions (3) cannot lose or gain a change of sign except when λ passes through a root of Δ_0. But as λ passes from $-\infty$ to $+\infty$, the functions (3) lose n changes of sign: hence the n roots of Δ_0 are real.

Also a change of sign once lost by the functions (3) as λ increases cannot be regained; for if a change of sign be regained it is clear that Δ_0 must have more than n roots. Thus, if k changes of sign are lost as λ increases from α to β, then Δ_0 must have exactly k roots between α and β.

§ 6. Let Δ'_r be the value of Δ_r when $\lambda = 0$, then it is clear that the conditions that all the roots of Δ_0 are negative, are that Δ'_0, Δ'_1, Δ'_2, ..., Δ'_n all have the same sign. Now $(-1)^m\Delta'_n$ is positive. Thus the conditions that all the roots are negative are

$$(-1)^m\Delta'_0,\ (-1)^m\Delta'_1,\ \dots,\ (-1)^m\Delta'_{n-1}$$

all positive. These are the conditions for the stability of the system. Δ'_0 may be formed by bordering the determinant $|b_{ij}|$ with the c's, and Δ'_1, Δ'_2, &c. are then formed by striking out a conjugate row and column, a second, and so on.

§ 7. In place of the determinants Δ_1, Δ_2, &c., another set of auxiliary functions may be used. Denoting the determinant in (1) by δ_0, let δ_r be the determinant formed by bordering δ_0 symmetrically with r additional rows of c's which may be taken arbitrarily subject to the conditions that the minors of order $m+r$ formed out of the border of δ_r are not all zero.

Then the functions

$$\delta_0,\ \delta_1,\ \delta_2,\ \dots,\ \delta_n \ \dots\dots\dots\dots\dots\dots\dots\dots(4)$$

may be used in precisely the same way as the functions (3). The quadric $\Sigma a_{ij}x_ix_j$ being one-signed and positive for all values of the x's which satisfy (1), is *a fortiori* one-signed and positive for all values of the x's which satisfy (1), and the r equations

$$\Sigma_i c_{i,\,m+s}x_i = 0 \ (s = 1, 2, \dots, r).$$

Hence, $\bar{\delta}_r$ denoting the value of δ_r when λ is replaced by unity and each b by zero, it is known (see preceding paper) that $(-1)^{m+r}\bar{\delta}_r$ is positive for all values of r from 1 to $n-1$. Also $\bar{\delta}_n = (-1)^l C^2$, where

$$C = |c_{ij}| \ (i, j = 1, 2, \dots, l).$$

Thus the coefficients of the highest power of λ in the functions (4) are alternately positive and negative or negative and positive. Thus, as λ increases from $-\infty$ to $+\infty$, the functions (4) gain n changes of sign, and the conditions that all the roots of δ_0 are negative are

$$(-1)^m\delta'_0,\ (-1)^{m+1}\delta'_1,\ \dots,\ (-1)^{m+n-1}\delta'_{n-1}$$

all positive.

If we make

$$c_{r,\,m+r} = 1,\ (r = 1, 2, \ldots, n),$$

and all the rest of the c's in the arbitrary bordering zero, we have $\delta_r = (-1)^r \Delta_r$. Thus the method of this section includes as a particular case the method of the preceding sections.

§ 8. The theorem that the roots of $|a_{ij}\lambda + b_{ij}|$ are all real when the quadric $\Sigma a_{ij} x_i x_j$ is one-signed and positive, is usually attributed to Lord Kelvin. See Thomson and Tait's *Natural Philosophy*, 1st edition, p. 279. It is therefore worthy of note that Sylvester had previously shewn that the roots of $|a_{ij}\lambda + b_{ij}|$ are all real, provided $D_0, D_1, \ldots, D_{l-1}$ are all positive or all negative where D_r is the discriminant of the quadric found by making $x_1, x_2, \ldots, x_r$ all zero in $\Sigma a_{ij} x_i x_j$. See *Philosophical Magazine*, Vol. VI., 1853, p. 214.

Melbourne,
***March* 21*st*, 1896.**

THE CONTENT OF THE COMMON SELF-CONJUGATE n-gon OF TWO n-ary QUADRICS.

By Professor *E. J. Nanson.*

1. Let $u, = \Sigma a_{ij} x_{ij}$, be an n-ary quadric, and let $P_1, \ldots, P_n$ be n points forming an n-gon self-conjugate with respect to u so that, if $x_{r1}, \ldots, x_{rn}$ are the coordinates of P_r, then

$$\Sigma a_{ij} x_{ri} x_{sj} = 0,\ (i, j, r, s = 1, 2, \ldots, n;\ r \neq s).$$

Let $\Delta = |a_{ij}|$, $P = |x_{ij}|$, and let u_r be the power of the point P_r with respect to the quadric u, that is the result of substituting the coordinates x_r of P_r for x in u, so that

$$u_r = \Sigma a_{ij} x_{ri} x_{rj},$$

then multiplying Δ twice by P, we readily find that

$$\begin{aligned} P^2\Delta &= u_1 u_2 \ldots u_n \\ &= \Pi u_r \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1). \end{aligned}$$

2. If $u', = \Sigma a'_{ij} x_i x_j$ be another n-ary quadric, then $\lambda u + u'$ is any quadric through the intersection of u, u'. If we determine λ by the condition

$$f(\lambda) \equiv |\lambda a_{ij} + a'_{ij}| = 0 \ \ldots\ldots\ldots\ldots\ldots (2),$$

then $\lambda u + u'$ will be a degenerate quadric having a singular point, and n such degenerate quadrics can be drawn. The

singular points of these n degenerate quadrics are the vertices of the n-gon self-conjugate to the two quadrics u, u'.

3. If, for a moment, we suppose u to be such a degenerate quadric the coordinates of the singular point of u are given by

$$\frac{x_i^2}{A_{ii}} = \frac{x_i x_j}{A_{ij}} = \frac{1}{K},$$

where A_{ii}, A_{ij} are the co-factors of a_{ii}, a_{ij} in Δ, and K is the determinant obtained by bordering Δ symmetrically with the coefficients of the identical equation

$$\Sigma b_i x_i = 1$$

which connects the coordinates x. Thus the power with respect to the quadric u' of the singular point of the degenerate quadric u is $(\Sigma A_{ij} a'_{ij})/K$.

Now $\Sigma A_{ij} a'_{ij}$ is the coefficient of μ in $| a_{ij} + \mu a'_{ij} |$. Hence we see that the power with respect to the quadric u of the singular point of the degenerate quadric $\lambda u + u'$ is θ / k, where θ is the coefficient of μ in

$$| \lambda a_{ij} + a'_{ij} + \mu a_{ij} | \text{ .... } (3),$$

and k is the bordered discriminant of $\lambda u + u'$, λ being any root of (2).

4. The determinant (3) is $f(\lambda + \mu)$, so that

$$\theta = f'(\lambda) \text{ .... } (4).$$

5. Let Δ', K' be the discriminant and bordered discriminant of u'. Then if $\Delta_1, \ldots, \Delta_{n-1}$ be the invariants intermediate to Δ, Δ', and $K_1, \ldots, K_{n-2}$ the invariants intermediate to K, K', we have

$$f(\lambda) = \Delta \lambda^n + \Delta_1 \lambda^{n-1} + \ldots + \Delta_{n-1} \lambda + \Delta_n,$$
$$k = K \lambda^{n-1} + K_1 \lambda^{n-2} + \ldots + K_{n-2} \lambda + K_{n-1},$$

where Δ_n, K_{n-1} are written for Δ', K'.

6. Now let $\lambda_1, \ldots, \lambda_n$ be the n roots of $f(\lambda)$ and $\theta_1, \ldots, \theta_n$ the corresponding values of θ, then we have from (4),

$$\Pi \theta_r = \Pi f'(\lambda_r)$$
$$= \epsilon \Delta^n \zeta \text{ .... } (5),$$

where $\epsilon = (-1)^{\frac{1}{2}n(n-1)}$ and ζ is the product of the squares of the differences of the roots $\lambda_1, \ldots, \lambda_n$.

7. Let R denote the dialytic eliminant

$$\begin{vmatrix} 0, & 0, & \dots, & \Delta, & \Delta_1, & \Delta_2, & \dots, & \Delta_n \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \Delta, & \Delta_1, & \dots, & \Delta_{n-2}, & \Delta_{n-1}, & \Delta_n, & \dots, & 0 \\ K, & K_1, & \dots, & K_{n-2}, & K_{n-1}, & 0, & \dots, & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0, & 0, & \dots, & 0, & K, & K_1, & \dots, & K_{n-1} \end{vmatrix}$$

of $f(\lambda)$ and k. Then the n values $k_1, \dots, k_n$ of k are the roots of the expression found by writing $K_{n-1} - k$ in place of K_{n-1} in R. Thus

$$\Pi k_r = \epsilon \Delta^{-n+1} R \quad \dots\dots\dots\dots\dots\dots\dots (6).$$

8. Let $P_1, \dots, P_n$ be taken to be the vertices of the common self-conjugate n-gon of u, u', then from section 3 and equations (1), (5), (6), we have

$$P^2\Delta = \Pi\theta_r / k_r$$
$$= R^{-1}\Delta^{2n-1}\zeta,$$

therefore $$P^2 = R^{-1}\Delta^{2n-2}\zeta.$$

9. If δ be the discriminant of $f(\lambda)$ written so that the coefficient of $\Delta^{n-1}\Delta'^{n-1}$ is $+1$, it may be proved that

$$\delta = \epsilon n^{-n}\Delta^{2n-2}\zeta.$$

Thus we have

$$P^2 = \epsilon n^n \delta R^{-1} \quad \dots\dots\dots\dots\dots\dots\dots (7).$$

10. If C be the content of the self-conjugate n-gon $P_1, \dots, P_n$ and C' the content of the n-gon of reference, we have

$$C = C' b_1 b_2 \dots b_n P.$$

Thus by (7), we obtain

$$C^2 = \epsilon n^n C'^2 b_1^2 b_2^2 \dots b_n^2 \delta R^{-1}.$$

If we give to n the values 3, 4 this formula gives an expression for the area of the common self-conjugate triangle of two conics, and for the volume of the common self-conjugate tetrahedron of two conicoids referred to any kind of point coordinates.

The expression for the area of the common self-conjugate triangle of two conics referred to rectangular axes given by Mr. Leudesdorf, Vol. VI., p. 151, follows at once from (7).

Melbourne, *May* 19, 1896.

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THE REDUCTION OF THE SECOND VARIATION OF AN INTEGRAL.

By *A. C. Dixon, Sc.D.*

1. THE object of the present paper is to simplify the process by which the second variation of an integral is generally reduced. The results agree with those of Clebsch (*Crelle's Journal*, Vols. LV. and LVI.), but there is a slight increase of generality, as in his papers the possibility of a function occurring in the subject of integration without its derivatives is not considered. There is also, I believe, some novelty in the treatment of equations of condition.

Three cases are considered, the first being the ordinary one with one dependent and one independent variable, the second that with several dependent variables and one independent, the third that of a multiple integral with several dependent variables.

Case I. One independent variable and one dependent.

2. Let the integral be $\int V dx$, limits being understood throughout the work. Suppose V a function of y and its first n derivatives $y_1, y_2, \ldots, y_n$.

Let the reduced form of $\delta \int V dx$ be $[Q] + \int v \delta y dx$, the first term being a linear function of the values of δy, δy_1, δy_2, ..., δy_{n-1} at the limits; for the purpose of the present reduction we may suppose that the limiting values of x are fixed.

3. Now let δ' denote the variation produced by a small change in the form of y other than δy, and suppose

$$\delta' \delta y = \delta \delta' y = 0,$$

so that neither variation of y is affected by the other. Then it is easily seen that

$$\begin{aligned}\delta' \delta \int V dx &= \delta \delta' \int V dx \\ &= \int \left(\sum_r \sum_s \frac{\partial^2 V}{\partial y_r \partial y_s} \delta' y_r \delta y_s \right) dx \begin{pmatrix} r = 0, 1, 2, \ldots, n \\ s = 0, 1, 2, \ldots, n \end{pmatrix},\end{aligned}$$

y_0 being understood to mean y.

But
$$\delta' \delta \int V dx = [\delta' Q] + \int \delta' v \delta y dx,$$
$$\delta \delta' \int V dx = [\delta Q'] + \int \delta v \delta' y dx,$$

if Q' denotes the value of Q when $\delta' y$ is put in the place of δy throughout.

Also
$$\delta v = \theta\,(\delta y),$$
$$\delta' v = \theta\,(\delta' y),$$

where θ denotes a certain linear differential operator of order $2n$, whose coefficients involve $x, y, y_1, \ldots$.

We have therefore, since $\delta'\delta\int V dx = \delta\delta'\int V dx$,

$$\int\{\delta y\theta(\delta' y) - \delta' y\theta(\delta y)\}\,dx = [\delta Q'] - [\delta' Q].$$

The operator θ is therefore such that the expression $u\theta t - t\theta u$ can be integrated of itself without the forms of t and u being assigned.

4. In this expression write uz for t, then the coefficient of z is $u\theta u - u\theta u$ or zero, so that in the integral the coefficient of z must be constant. But now every coefficient in the integral must be of the second degree in the arbitrary function u and its derivatives, so that this constant must be zero. The integral is therefore ϕz_1, ϕ being a new linear differential operator of order $2n-2$. The arbitrary constant of integration is ignored.

5. We may now follow Bertrand (see Todhunter's *History*, p. 263) in his proof that the operator ϕ has the same nature as θ. Let w be another arbitrary function, then

$$w_1\phi z_1 - z_1\phi w_1 = \frac{d}{dx}\,\{w\phi z_1 - z\phi w_1\} - w\,\frac{d}{dx}\,\phi z_1 + z\,\frac{d}{dx}\,\phi w_1$$

$$= \frac{d}{dx}\,\{w\phi z_1 - z\phi w_1\} - w\,\{u\theta(uz) - uz\theta u\} + z\,\{u\theta(uw) - uw\theta u\}$$

$$= \frac{d}{dx}\{w\phi z_1 - z\phi w_1\} - (uw)\,\theta(uz) + (uz)\,\theta(uw).$$

Hence $w_1\phi z_1 - z_1\phi w_1$ is integrable of itself without the assignment of particular forms to z_1, w_1, for by hypothesis this is true of

$$(uz)\,\theta(uw) - (uw)\,\theta(uz).$$

There is thus no need to shew that θ has any particular form, as is generally done; all that is necessary to enable us to reduce the second variation by successive integrations by parts in the usual way has now been proved.

Case II. Several dependent variables.

6. The case when there are any number of variables, dependent and independent, and when there are also equations of condition, the order of multiplicity of the integrals being of course the same as the number of independent variables, may be treated in the same way.

7. If V is the subject of integration, and $\phi_1 = 0$, $\phi_2 = 0$ ... are the equations of condition, it is noticeable that the differential equations giving the solution are the same as if

$$V + \lambda_1\phi_1 + \lambda_2\phi_2 + \ldots$$

were the subject and λ_1, λ_2 ... were considered as additional dependent variables, there being now no equations of condition.

8. If we compare the second variations in the two cases, we see that the expression for the latter must reduce to that for the former by means of the equations of condition and the conditions, satisfied by the variations, which they give; in fact, that the equations of condition are satisfied by the functions sought, both before and after variation, and that the second (or any other) variation of the integral of V is the same as that of the integral of $V + \lambda_1\phi_1 + \lambda_2\phi_2 + \ldots$ for all variations that are *permissible in the former* (*restricted*) *case.*

The variations of λ_1, λ_2 ... disappear if the equations of condition are satisfied, so that it does not matter whether they are supposed to vary or not.

9. By the use of equations of condition we can ensure that no differential coefficients above the first enter the subject of integration.

Thus we have to consider the second variation of the integral of a function involving an arbitrary number of unknown functions, and the first derivatives of some of them; the integral may be single or multiple, and there is an important difference in the two cases.

10. Consider in the first place a single integral in which the derivatives of all the functions occur.

Let y_1, y_2, ..., y_n be the functions, x the independent variable, $\int V dx$ the integral whose stationary value is being investigated.

F 2

Let the reduced form of $\delta\int V dx$ be

$$[Q] + \int(\sum_{r=1}^{n} v_r \delta y_r)\, dx.$$

As before, if δ' denote any variation, independent of δ,

$$\Sigma(\delta' v_r \delta y_r - \delta v_r \delta' y_r)$$

is integrable of itself.

We have to transform $\int\Sigma(\delta v_r \delta y_r)\, dx$; the process will apply to $\int\Sigma(\delta' v_r \delta y_r)\, dx$.

11. δv_r may be expressed as $\sum_{s=1}^{n} \theta_{rs} \delta y_s$, θ_{rs} being a linear differential operator of the second order. Then

$$\delta' v_r = \sum_{s=1}^{n} \theta_{rs} \delta' y_s,$$

and, by supposing all the variations but δy_r and $\delta' y_s$ to vanish, we find that

$$z\theta_{rs} u - u\theta_{sr} z$$

is integrable of itself.

Let $z_s = \alpha_s$ $(s = 1, 2, \ldots, n)$ be a solution of the equations

$$\sum_{s=1}^{n} \theta_{rs} z_s = 0 \ (r = 1, 2, \ldots, n).$$

Let $z_s = \beta_s$, $z_s = \gamma_s$, ..., $z_s = \kappa_s$ be $(n-1)$ other independent solutions, and put

$$\delta y_s = A\alpha_s + B\beta_s + C\gamma_s + \ldots + K\kappa_s \ (s = 1, 2, \ldots, n),$$

$$\delta' y_s = A_1\alpha_s + B_1\beta_s + C_1\gamma_s + \ldots + K_1\kappa_s \ (s = 1, 2, \ldots, n),$$

which will clearly be possible if the determinant of the n solutions does not vanish.

The expression to be transformed is the integral of the sum of such expressions as

$$\sum_{r=1}^{n} \sum_{s=1}^{n} A\alpha_r \theta_{rs} (B_1\beta_s).$$

12. Take this one by itself; it is equal to

$$A\{\Sigma\Sigma\alpha_r\theta_{rs}(B_1\beta_s) - \Sigma\Sigma B_1\beta_r\theta_{rs}\alpha_s\}.$$

The expression inside the brackets can be integrated, and will thus give

$$\{\alpha, B_1\beta\}$$

if we write $\{u, w\}$ for the integral of $\Sigma\Sigma u_r\theta_{rs}w_s - \Sigma\Sigma w_r\theta_{rs}u_s$.

We shall suppose the constant of integration in the expression $\{v, w\}$ taken as zero, so that there is no absolute term.

Every term in $\{\alpha, B_1\beta\}$ must contain B_1 or $\frac{dB_1}{dx}$, and, by putting $B_1 = 1$, we find that the coefficient of B_1 is $\{\alpha, \beta\}$. Assume this to be zero. Then what is left after integrating by parts is $\frac{dA}{dx}\frac{dB_1}{dx}$ multiplied by a known function of x.

13. Now as to the assumption that $\{\alpha, \beta\}$ is zero—clearly its differential coefficient is zero, so that it is constant. In fact, if $z_s = u_s$, $z_s = w_s$ are any two systems of solutions $\{u, w\}$ is a constant, and

$$\{w, u\} = -\{u, w\}.$$

Now it is known that the whole system of solutions is found by differentiating the solution of the system $v_r = 0$ with respect to the constants involved, and that there are $2n$ independent solutions of the system $\sum_s \theta_{rs}z_s = 0$. Of these, $2n - 2$ other than $z_s = \alpha_s$ will satisfy the condition

$$\{\alpha, z\} = 0.$$

For, if $\qquad \{\alpha, u\} = f, \ \{\alpha, w\} = g,$

we have $\{\alpha, gu - fw\} = 0$; this process enables us to ensure that all but one of the $2n - 1$ integrals satisfy the condition.

Take one of the $2n - 2$ systems as the system β, and we can in the same way ensure that all but one of the $2n - 3$ that are left will satisfy the condition

$$\{\beta, z\} = 0.$$

Take any of these $2n - 4$ as the system γ. It is clear that in this way we may successively form just n systems $\alpha, \beta, \gamma, \ldots, k$. Evidently, also $\{\alpha, \alpha\} = 0$, $\{\beta, \beta\} = 0$. The assumption then is justified.

14. Thus the part of the expression $\int\Sigma(\delta' v_r \delta y_r)\, dx$ under the integral sign is reduced to an expression linear and homogeneous in $\frac{dA}{dx}, \frac{dB}{dx} \ldots$ and also in $\frac{dA_1}{dx}, \frac{dB_1}{dx} \ldots$, and

in the second variation this becomes a homogeneous quadratic in the n quantities $\frac{dA}{dx}, \frac{dB}{dx} \dots$.

15. Again, as to the coefficients in this expression—the coefficient of $\frac{d^2}{dx^2}$ in θ_{rs} is $-\frac{\partial^2 V}{\partial y_r' \partial y_s'}$, dashes denoting x-differentiation.

The coefficient of B_1' in $\{\alpha, B_1\beta\}$ is then

$$-\Sigma\Sigma\alpha_r\beta_s \frac{\partial^2 V}{\partial y_r' \partial y_s'},$$

and that of $A'B_1'$ in the final expression is the same quantity with the positive sign.

Hence the final expression for the second variation consists of terms at the limits together with the integral of

$$\sum_{r=1}^{n} \sum_{s=1}^{n} p_r p_s \frac{\partial^2 V}{\partial y_r' \partial y_s'},$$

where p_r denotes $A'\alpha_r + B'\beta_r + C'\gamma_r + \dots + K'\kappa_r$.

16. Suppose now that V contains other functions $y_{n+1}, y_{n+2}, \dots, y_{n+m}$, without their derivatives. The process will apply, with a few modifications. The summations, &c. must now be taken from 1 to $n+m$, θ_{rs} will be an operator of the first order if one and only one of its suffixes $> n$, and of order zero if both $> n$. In the latter case it is a simple function of x, by which the operand is to be multiplied.

17. As before, take n particular solutions, and put

$$\delta y_s = A\alpha_s + B\beta_s + \dots + K\kappa_s, \ (s = 1, 2, \dots, n),$$

$$\delta y_s = A\alpha_s + B\beta_s + \dots + K\kappa_s + \lambda_s, \ (s = n+1, n+2, \dots, n+m).$$

For convenience suppose that $\lambda_s = 0$, $(s = 1, 2, \dots, n)$.

The transformation of the expression

$$\Sigma\Sigma A\alpha_r\theta_{rs}(B\beta_s)$$

follows as before, with the choice of the solutions so that $\{\alpha, \beta\} = 0$, &c. The whole number of solutions is $2n + m$.

18. We have also to consider the expressions

$$\Sigma\Sigma A\alpha_r\theta_{rs}(\lambda_s), \ \Sigma\Sigma\lambda_r\theta_{rs}(A\alpha_s), \ \Sigma\Sigma\lambda_r\theta_{rs}(\lambda_s),$$

in which it is to be remembered that the quantities λ are unrestricted in form, except those that vanish.

The first and second of these three expressions are for our purpose equivalent, their difference being integrable. Also the first may be written $A\frac{d}{dx}\{\alpha, \lambda\}$, so that it is equivalent to $-\{\alpha, \lambda\}A'$.

The factor $\{\alpha, \lambda\}$ is a homogeneous linear function of the λ's, for its differential coefficient contains no second derivatives of $\lambda_{n+1}, \lambda_{n+2}, \ldots, \lambda_{n+m}$.

Also $\Sigma\Sigma\lambda_r\theta_{rs}(\lambda_s)$ is not a differential expression at all.

19. Hence the expression under the integral sign in the second variation is reduced to a homogeneous quadratic function of $A', B', C', \ldots, K', \lambda_{n+1}, \lambda_{n+2}, \ldots, \lambda_{n+m}$. The part involving $A', B', \ldots, K'$ only is found as before to be

$$\sum_{r=1}^{n}\sum_{s=1}^{n} p_r p_s \frac{\partial^2 V}{\partial y_r' \partial y_s'}.$$

The coefficient of $\frac{d}{dx}$ in $\theta_{n+1,1}$ or $-\theta_{1,n+1}$ is $\frac{\partial^2 V}{\partial y_{n+1}\partial y_1'}$, and so in other like cases. Hence the coefficient of $\lambda_{n+1}A'$ in the final quadratic expression is

$$2\sum_{r=1}^{n}\alpha_r \frac{\partial^2 V}{\partial y_{n+1}\partial y_r'}.$$

We see then that the whole expression is

$$\sum_{r=1}^{n}\sum_{s=1}^{n} p_r p_s \frac{\partial^2 V}{\partial y_r' \partial y_s'} + \sum_{r=n+1}^{n+1}\sum_{s=n+m}^{n+m} \lambda_r \lambda_s \frac{\partial^2 V}{\partial y_r \partial y_s} + 2\sum_{r=1}^{n}\sum_{s=n+1}^{n+m} p_r \lambda_s \frac{\partial^2 V}{\partial y_r' \partial y_s},$$

a quadratic function of $n+m$ quantities.

20. Suppose now that there are equations of condition connecting the dependent variables. In accordance with what was said above (§§ 7, 8) we may take $v_{n+1}=0$ as one of these equations, and y_{n+1} as the corresponding multiplier; y_{n+1} will then not occur elsewhere in V.

$\delta v_{n+1}=0$ will be a condition connecting the variations that are permissible, and will be a linear differential equation of the first order. If we make the substitution

$$\delta y_s = A\alpha_s + B\beta_s + \ldots + K\kappa_s + \lambda_s,$$

the coefficients of $A, B, \ldots, K$ will vanish since the substitutions $\delta y_s = \alpha_s, \beta_s \ldots$ satisfy $\delta v_{n+1} = 0$. This equation is indeed one of the system of which they are assumed solutions.

Hence an equation of condition leads to a linear relation connecting the derivatives of $A, B, C, \ldots, K$ and the quantities λ, and as before we may introduce $p_1, p_2, \ldots, p_n$ instead of $A', B', \ldots, K'$.

21. It is easily seen that this relation is

$$\sum_{r=1}^{n} p_r \frac{\partial^2 V}{\partial y_{n+1} \partial y_r} + \sum_{s=1}^{m} \lambda_{n+s} \frac{\partial^2 V}{\partial y_{n+1} \partial y_{n+s}} = 0,$$

the left-hand side being half the coefficient of λ_{n+1} in the quadratic form, since $\dfrac{\partial^2 V}{\partial y^2_{n+1}} = 0$.

Hence, if $v_{n+1} = 0$ is an equation of condition, and y_{n+1} its multiplier, λ_{n+1} and y_{n+1} disappear from the final quadratic form, and the variables that are left in the form are subject to a linear relation. The same will be true, if there are more conditions than one, for each of them.

Case III. Multiple integrals.

22. In treating a multiple integral with respect to r variables $x_1, x_2, \ldots, x_r$ we shall use SV to denote $\iiint \ldots V dx_1 dx_2 \ldots dx_r$ taken over the whole range under consideration, and $S_1, S_2, \ldots$ to denote integration over the boundary of that range, the number of integrations being $(r-1)$ and the suffix shewing which of the differentials is left out. Thus

$$S\left(\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \ldots\right) = S_1(X_1) + S_2(X_2) + \ldots .$$

An integrable expression will mean one of the form

$$\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \ldots,$$

and the formula for integration by parts is

$$S\left(U \frac{\partial V}{\partial x_t}\right) = S_t(UV) - S\left(V \frac{\partial U}{\partial x_t}\right).$$

23. *Lemma.* It is useful to note that if the r functions $X_1, X_2, \ldots$ are such that

$$\sum_{t=1}^{r} \frac{\partial X_t}{\partial x_t} = 0,$$

we may express them thus

$$X_t = \sum_{s=1}^{r} \frac{\partial Y_{ts}}{\partial x_s},$$

where $Y_{ts} = - Y_{st}$, $Y_{tt} = 0$.

The expression is by no means unique; we may clearly write

$$Y_{ts} + \frac{\partial \mu}{\partial x_r}, \quad Y_{tr} - \frac{\partial \mu}{\partial x_s}, \quad Y_{st} - \frac{\partial \mu}{\partial x_r}, \quad Y_{sr} + \frac{\partial \mu}{\partial x_t}, \quad Y_{rt} + \frac{\partial \mu}{\partial x_s}, \quad Y_{rs} - \frac{\partial \mu}{\partial x_t},$$

in place of Y_{ts}, Y_{tr}, Y_{st}, Y_{sr}, Y_{rt}, Y_{rs} respectively, without making any difference, whatever function μ may be.

To prove this lemma, choose Y_2, Y_3, ..., so that

$$X_r = \frac{\partial Y_r}{\partial x_1} \; (r = 2, 3, \ldots).$$

Then
$$\frac{\partial X_1}{\partial x_1} + \Sigma \frac{\partial^2 Y_r}{\partial x_1 \partial x_r} = 0,$$

so that $X_1 + \Sigma \frac{\partial Y_r}{\partial x_r}$ is independent of x_1. Suppose it to be $\frac{\partial Y}{\partial x_2}$, Y being also independent of x_1, then we may take $Y_2 - Y$ instead of Y_2, that is to say, we may assume $Y = 0$.

Then we have

$$Y_{1r} = - Y_{r1} = - Y_r,$$
$$Y_{rs} = 0 \, (r = 2, 3, \ldots, s = 2, 3, \ldots).$$

24. Now let $S(V)$ be the integral whose variation is under discussion. Suppose V to involve $n + m$ functions $y_1, y_2, \ldots, y_{n+m}$ and first derivatives of $y_1, y_2, \ldots, y_n$. Let us write $y^{(s)}$ for $\frac{\partial y}{\partial x_s}$.

We know that

$$\delta S(V) = S\left(\sum_{t=1}^{n+m} v_t \delta y_t\right) + Q,$$

where Q is a boundary integral and v_t is formed from V by the usual rule.

As before, we have

$$\delta' \delta S(V) = \delta \delta' S(V),$$

and thus
$$S(\Sigma\delta' v\delta y_t - \Sigma\delta v_t\delta' y_t)$$
is equal to a boundary integral, so that
$$\Sigma\delta' v_t\delta y_t - \Sigma\delta v_t\delta' y_t$$
is an integrable expression.

We may again write
$$\delta v_t = \sum_{s=1}^{n+m} \theta_{ts}\delta y_s,$$
where θ_{ts} is a partial differential operator of order not higher than the second, and we again find that $z\theta_{ts}u - u\theta_{st}z$ is an integrable form.

25. Take n linearly independent solutions of the equations
$$\sum_{s=1}^{m+n} \theta_{ts}z_s = 0\ (t = 1,\ 2,\ \dots,\ n+m).$$
Denote them by $z_s = \alpha_s$, $z_s = \beta_s$, ..., $z_s = \kappa_s$, and put
$$\delta y_s = A\alpha_s + B\beta_s + \dots + K\kappa_s\ (s = 1,\ 2,\ \dots,\ n),$$
$$\delta y_s = A\alpha_s + B\beta_s + \dots + K\kappa_s + \lambda_s\ (s = n+1,\ n+2,\ \dots,\ n+m),$$
$$\lambda_s = 0,\ (s = 1,\ 2,\ \dots,\ n).$$

Then consider the expression
$$S\sum_{t=1}^{n+m}\sum_{s=1}^{n+m} A\alpha_t\theta_{ts}(B\beta_s),$$
which is part of $\delta^2 SV$.

It may be written
$$SA\{\Sigma\Sigma\alpha_t\theta_{ts}(B\beta_s) - \Sigma\Sigma B\beta_t\theta_{ts}(\alpha_s)\}.$$

The expansion within brackets is integrable; let the integral be
$$S_1(\zeta_1 B) + S_2(\zeta_2 B) + S_3(\zeta_3 B) + \dots,$$
ζ_1, ζ_2, ... denoting linear differential operators of the first order. Then the residue, after integrating by parts, is
$$-S(A^{(1)}\zeta_1 B + A^{(2)}\zeta_2 B + A^{(3)}\zeta_3 B + \dots).$$

The expression to be integrated here contains B undifferentiated. We do not, however, suppose the coefficient of B to be zero, but transform the terms by means of the lemma proved above.

26. The coefficient of B is

$$A^{(1)}\zeta_1 1 + A^{(2)}\zeta_2 1 + A^{(3)}\zeta_3 1 + \dots .$$

Now

$$\Sigma \frac{\partial}{\partial x_t}(\zeta_t B) = \Sigma\Sigma \alpha_t \theta_{ts}(B\beta_s),$$

so that

$$\Sigma \frac{\partial}{\partial x_t}(\zeta_t 1) = 0.$$

We may then suppose that

$$\zeta_t 1 = \sum_{s=1}^{r} \frac{\partial \omega_{ts}}{\partial x_s} \; (t = 1, 2, \dots, r),$$

where $\omega_{ts} = -\omega_{st}$, and $\omega_{tt} = 0$.

Take the terms involving ω_{12}. We have

$$\begin{aligned} &S(BA^{(1)}\omega_{12}{}^{(2)} - BA^{(2)}\omega_{12}{}^{(1)}) \\ &= S_2(B\omega_{12}A^{(1)}) - S_1(B\omega_{12}A^{(2)}) \\ &\quad - S(B^{(2)}\omega_{12}A^{(1)} - B^{(1)}\omega_{12}A^{(2)}), \end{aligned}$$

the terms in $\dfrac{\partial^2 A}{\partial x_1 \partial x_2}$ disappearing.

Hence the terms that contain B give a boundary integral together with the r-ple integral of an expression consisting of such terms as

$$\omega_{st} \frac{\partial(B, A)}{\partial(x_s, x_t)}.$$

27. The whole expression $S\Sigma\Sigma A\alpha_t\theta_{ts}(B\beta_s)$ has thus been reduced to a boundary integral together with the S-integral of an expression homogeneous and of the second degree in the partial derivatives of A and B.

The other typical expressions to be considered are

$$\Sigma\Sigma\lambda_t\theta_{ts}(A\alpha_s),\; \Sigma\Sigma A\alpha_t\theta_{ts}\lambda_s,\; \Sigma\Sigma\lambda_s\theta_{ts}\lambda_t.$$

The last is simply a quadratic function of $\lambda_{n+1}, \dots, \lambda_{n+m}$ without their derivatives. The two former are equivalent, since their difference is integrable. Let us again consider the second. It may be written

$$A\Sigma\Sigma\{\alpha_t\theta_{ts}\lambda_s - \lambda_s\theta_{st}\alpha_t\},$$

so that the second factor is integrable, and the residue after integrating by parts is linear and homogeneous in the derivatives of A, as also in the quantities λ.

28. Thus the whole expression subject to the operation S has been reduced to a homogeneous quadratic function of the derivatives of $A, B, C, \ldots, K$ and the quantities λ themselves.

29. In order to find the coefficients in this quadratic expression, let us go back to the identity

$$\delta'\delta SV = \delta\delta' SV,$$

or

$$S[\Sigma\Sigma\delta y_\iota \theta_{\iota s} \delta' y_s - \Sigma\Sigma\delta' y_s \theta_{s\iota} \delta y_\iota] = \delta Q' - \delta' Q$$

$$= \Sigma\Sigma S_\iota \left[\delta \frac{\partial V}{\partial y_s^{(\iota)}} \delta' y_s - \delta' \frac{\partial V}{\partial y_s^{(\iota)}} \delta y_s\right].$$

Putting $\delta y_s = \alpha_s$, $\delta' y_s = B\beta_s$, we find that

$$\zeta_i B = -\Sigma\Sigma\Sigma \frac{\partial^2 V}{\partial y_s^{(i)} \partial y_\iota^{(j)}} \alpha_s \beta_\iota B^{(j)}$$

$$+ B\Sigma\Sigma \frac{\partial^2 V}{\partial y_s^{(i)} \partial y_\iota} (\alpha_\iota \beta_s - \alpha_s \beta_\iota)$$

$$+ B\Sigma\Sigma\Sigma \frac{\partial^2 V}{\partial y_s^{(i)} \partial y_\iota^{(j)}} (\alpha_\iota^{(j)} \beta_s - \alpha_s \beta_\iota^{(j)}).$$

The coefficient of B here is $\zeta_i 1$.
For ω_{i1} we take $\int \zeta_i 1 dx_1$, adjusting the arbitrary constants (which are functions of $x_2, x_3, \ldots, x_r$) so that

$$\Sigma \frac{\partial \omega_{i1}}{\partial x_i} = -\zeta_1 1.$$

When neither suffix is 1, we put $\omega_{ij} = 0$.
Let us denote these values of the quantities $\omega_{ij} \ldots$ by $(\alpha, \beta)_{ij}, \ldots$, so that $(\beta, \gamma)_{ij}$, &c. will have a like meaning.
Then $(\alpha, \beta)_{ij} = -(\beta, \alpha)_{ij}$, $(\alpha, \alpha)_{ij} = 0$.

30. The coefficient of $A^{(i)}B^{(j)}$ in $-\Sigma A^{(i)}\zeta_i B$ is

$$\Sigma\Sigma \frac{\partial^2 V}{\partial y_s^{(i)} \partial y_\iota^{(j)}} \alpha_s \beta_\iota.$$

This same term will be repeated through the interchange of A and B, α and β. The above coefficient must therefore be doubled.

In the same way the coefficient of $A^{(i)}A^{(j)}$ is

$$2\Sigma\Sigma\frac{\partial^2 V}{\partial y_s^{(i)}\partial y_t^{(j)}}\alpha_s\alpha_t,$$

and that of $A^{(i)2}$ is

$$\Sigma\Sigma\frac{\partial^2 V}{\partial y_s^{(i)}\partial y_t^{(i)}}\alpha_s\alpha_t.$$

Collecting the terms in which $\frac{\partial^2 V}{\partial y_s^{(i)}\partial y_t^{(j)}}$ appears, we find that their sum, so far, is

$$2\frac{\partial^2 V}{\partial y_s^{(i)}\partial y_t^{(j)}}(A^{(i)}\alpha_s + B^{(i)}\beta_s + \ldots)(A^{(j)}\alpha_t + B^{(j)}\beta_t + \ldots).$$

Let us write

$$A^{(i)}\alpha_s + B^{(i)}\beta_s + \ldots + K^{(i)}k_s = p_s^{(i)} \begin{pmatrix} s=1, 2, \ldots, n \\ i=1, 2, \ldots, r \end{pmatrix}^*,$$

and we have thus the aggregate of terms

$$\Sigma\Sigma\Sigma\Sigma\frac{\partial^2 V}{\partial y_s^{(i)}\partial y_t^{(j)}}p_s^{(i)}p_t^{(j)}\begin{pmatrix} s=1, 2, \ldots, n \\ t=1, 2, \ldots, n \end{pmatrix}\begin{pmatrix} i=1, 2, \ldots, r \\ j=1, 2, \ldots, r \end{pmatrix}.$$

31. To this aggregate must be added that of the terms such as

$$(A^{(i)}B^{(j)} - A^{(j)}B^{(i)})(\alpha, \beta)_{ij},$$

each of which again occurs twice.

Now $A^{(i)}$, $B^{(i)}$, ... can be expressed linearly in terms of $p_1^{(i)}, p_2^{(i)}, \ldots$; in fact

$$A^{(i)}\Delta = \Sigma a_s p_s^{(i)}\ (s=1, 2, \ldots, n),$$

where Δ is the determinant of the quantities

$$\alpha_s, \beta_s, \gamma_s, \ldots, (s=1, 2, \ldots, n),$$

---

* Here $p_s^{(i)}$ does not denote the derivative of a function p_s with respect to x_i.

and a_s is the minor of α_s. If we write $(ab)_{st}$ for the minor of α_s and β_t, we find

$$A^{(i)}B^{(j)} - A^{(j)}B^{(i)} = \Sigma (ab)_{st} (p_s^{(i)} p_t^{(j)} - p_s^{(j)} p_t^{(i)}) \div \Delta.$$

Hence the quantity to be added to the coefficient of $2p_s^{(i)} p_t^{(j)}$ is

$$\frac{1}{\Delta} \{(\alpha, \beta)_{ij} (ab)_{st} + (\alpha, \gamma)_{ij} (ac)_{st} + (\beta, \gamma)_{ij} (bc)_{st} + \ldots\}.$$

32. The coefficient of λ^2_{n+s} is $\frac{\partial^2 V}{\partial y^2_{n+s}}$, and that of $\lambda_{n+s}\lambda_{n+t}$ is $2\frac{\partial^2 V}{\partial y_{n+s} \partial y_{n+t}}$.

We have to find the coefficient of $\lambda_{n+s} A^{(i)}$. Let us put $\delta y_s = \alpha_s$, $\delta' y_s = \lambda_s$ and we shall have

$$S\Sigma\Sigma [\alpha_t \theta_{ts} \lambda_s - \lambda_s \theta_{st} \alpha_t] = -\Sigma\Sigma\Sigma S_i \left[\alpha_s \lambda_t \frac{\partial^2 V}{\partial y_s^{(i)} \partial y_t}\right].$$

The coefficient sought is therefore

$$2\Sigma \alpha_t \frac{\partial^2 V}{\partial y_t^{(i)} \partial y_{n+s}} \quad (t = 1, 2, \ldots, n).$$

The aggregate of these terms is

$$\Sigma\Sigma\Sigma \lambda_{n+s} p_t^{(i)} \frac{\partial^2 V}{\partial y_t^{(i)} \partial y_{n+s}} \begin{Bmatrix} i = 1, 2, \ldots, r \\ s = 1, 2, \ldots, m \\ t = 1, 2, \ldots, n \end{Bmatrix}.$$

33. Thus the second variation of a multiple integral is reduced to a boundary integral together with the S-integral of a homogeneous quadratic expression in the quantities $p_s^{(i)}$ and λ_s, in all $nr + m$ in number. The coefficients in this quadratic are generally second derivatives of V, but some of them have to be altered. The alteration need only take place on those which multiply such products as $p_s^{(i)} p_t^{(1)}$, where $i \neq 1$ and $s \neq t$, and the sum of the coefficients of $2p_s^{(i)} p_t^{(1)}$ and $2p_t^{(i)} p_s^{(1)}$ is still

$$\frac{\partial^2 V}{\partial y_s^{(i)} \partial y_t^{(1)}} + \frac{\partial^2 V}{\partial y_t^{(i)} \partial y_s^{(1)}}.$$

This agrees with Clebsch's results (*Crelle*, LVI., pp. 146-7).

34. We may see, just as in the case of one independent variable, that if there are equations of condition they may be taken as among the equations

$$v_{n+s} = 0,$$

that the corresponding multipliers and their variations disappear from the quadratic form, and that as many linear relations connecting the rest of the variables of the form are left.

35. It is also pointed out by Clebsch, following Jacobi, that the systems of integrals of the equations

$$\delta v_s = 0,$$

which are needed for the reduction of the second variation, may be found by varying the arbitrary constants and functions which enter into the complete solution of the system $v_s = 0$.

Queen's College,
Galway,
June, 1896.

ON CERTAIN DISTINCTIONS BETWEEN THE THEORIES OF CONVERGING FRACTIONS AND CONVERGING MULTIPLES.

By *Percy J. Heawood.*

THE application of the methods of continued fractions to the finding of approximate common multiples of two numbers is well known; but certain distinctions between the theory of a series of fractions $\frac{p}{q}$, ... converging to a given value $\frac{a}{b}$, and that of a series of pairs of multiples pb, qa, ... of a and b, continually approximating to each other, do not seem to be sufficiently noticed.

To begin with, the proposition that, $\frac{p}{q}$, ... being the successive convergents to $\frac{a}{b}$, the successive difference $pb \sim qa$, ... *do* continually diminish, goes further than the ordinary statement that the successive convergents continually approach nearer to $\frac{a}{b}$, although sufficiently obvious as being involved in the usual proof of it: which is in effect that, if $\frac{p}{q}$, $\frac{p'}{q'}$ be

any two consecutive convergents, we have $\frac{a}{b} = \frac{Mp'+p}{Mq'+q}$, and therefore $\frac{p'b-q'a}{qa-pb} = \frac{1}{M}$, where M is greater than 1: so that $p'b \sim q'a < pb \sim qa$. (We may, indeed, state the same fact in terms of the fractions by saying that not only do $\frac{p}{q}$, ... continually approach $\frac{a}{b}$, but that their differences from it diminish *more rapidly than their denominators increase*).

Again, if $\frac{r}{s}$ be the fraction nearest to $\frac{a}{b}$ with denominator s, less than q', (not $= q$ or nq)*; we have

$$\frac{\frac{r}{s} \sim \frac{p}{q}}{\frac{p'}{q'} \sim \frac{p}{q}} = \frac{q'}{s}(rq \sim sp) \;\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\frac{\frac{r}{s} \sim \frac{p'}{q'}}{\frac{p'}{q'} \sim \frac{p}{q}} = \frac{q}{s}(rq' \sim sp') \;\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

of which (1) gives the usual proof that *either* $\frac{r}{s}, \frac{p'}{q'}, \frac{a}{b}, \frac{p}{q}$, *or else* $\frac{p'}{q'}, \frac{a}{b}, \frac{p}{q}, \frac{r}{s}$ are in order of magnitude (ascending or descending); but we have further in either case,

$$\frac{\frac{r}{s} \sim \frac{a}{b}}{\frac{p'}{q'} \sim \frac{a}{b}} > \frac{\frac{r}{s} \sim \frac{p}{q}}{\frac{p'}{q'} \sim \frac{p}{q}}, \text{ therefore } > \frac{q'}{s} \text{ by (1)},$$

$$\frac{\frac{r}{s} \sim \frac{a}{b}}{\frac{p}{q} \sim \frac{a}{b}} > \frac{\frac{r}{s} \sim \frac{p'}{q'}}{\frac{p}{q} \sim \frac{p'}{q'}}, \text{ therefore } > \frac{q}{s} \text{ by (2)}.$$

---

* If s did $= nq$, r would $= np$; for $\frac{np}{nq} \sim \frac{a}{b} < \frac{p}{q} \sim \frac{p'}{q'} < \frac{1}{qq'} < \frac{q^{-1}}{nq}$ a fortiori, nq being less than q'; so that a difference of 1 in the numerator would take $\frac{np}{nq}$ further from $\frac{a}{b}$, (unless indeed $q = 1$). Rejecting this case, where $\frac{r}{s} = \frac{p}{q}$, $rq \sim sp$ must $= 1$ at least. It is obviously superfluous to consider any fraction $\frac{r}{s}$ not in its lowest terms.

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These add nothing, as far as the fractions are concerned, to the fact which is obvious already that $\frac{r}{s}$ is further from $\frac{a}{b}$ than $\frac{p'}{q'}$ is, but they prove

(1) $p'b \sim q'a < rb \sim sa$ which is more than the above result.

(2) $pb \sim qa < rb \sim sa$ the result corresponding to which for the fractions does not hold. This is important, as it proves that not only do each pair of multiples in the series approximate more nearly to each other than any lower pair of multiples whatever, but also that the series of multiples is complete, in a sense in which the series of converging fractions is not complete, in that there can be no intermediate multiples of even an intermediate degree of approximation.

For the series of fractions it is otherwise, as $\frac{r}{s}$ may be nearer to $\frac{a}{b}$ than $\frac{p}{q}$ is, when s lies between q and q'. However, it will be seen, by equation (1) above, that this cannot happen if $rq \sim sp$ be as much as 2 $\left(\text{for then } \frac{r}{s} \text{ will be more than twice as far from } \frac{p}{q} \text{ as } \frac{p'}{q'} \text{ is, and a fortiori further than } \frac{p}{q} \text{ from } \frac{a}{b}\right)$. We have the result that for such an intermediate fraction $rq - sp = \pm 1$; from which it follows at once that $\frac{p}{q}$ must be one of the two convergents which we may consider as immediately preceding $\frac{r}{s}$ if it be reduced to a continued fraction*; for otherwise we have $xs - yr = ps - qr$, where $\frac{x}{y}$ is such a convergent, which gives $\frac{x-p}{y-q} = \frac{r}{s}$; and this is impossible unless $x = p$ and $y = q$ (x, p being each less than r, and y, q than s, and r, s having no common measure). This seems the simplest way of proving that such a fraction must be of the form $\frac{mp + \mathrm{p}}{mq + \mathrm{q}}$, where $\frac{\mathrm{p}}{\mathrm{q}}$ is the convergent immediately preceding $\frac{p}{q}$. It is not, however, the case that *any* fraction

* A fraction may be reduced to a continued fraction with either an odd or an even number of convergents at pleasure, by taking the final quotient $=$, or not $=$ 1, as the case may require. The two series of convergents differ only in the presence or absence of an "extra" convergent just before the final one: hence the two which may be considered as occupying the position of last but one, referred to above. For one $rq - sp = +1$, for the other -1.

of this form is nearer to $\frac{a}{b}$ than $\frac{p}{q}$ is, and the point to be specially noticed here is the very convenient form in which the criterion may be put for distinguishing which of these are really of intermediate accuracy. The necessary and sufficient condition that the fraction $\frac{mp_{n-1}+p_{n-2}}{mq_{n-1}+q_{n-2}}$ may be nearer than $\frac{p_{n-1}}{q_{n-1}}$ is to $\frac{a}{b}$, *i.e.* to $\frac{Mp_{n-1}+p_{n-2}}{Mq_{n-1}+q_{n-2}}$ is plainly:

$$\frac{mp_{n-1}+p_{n-2}}{mq_{n-1}+q_{n-2}} \sim \frac{p_{n-1}}{q_{n-1}} < 2\left(\frac{Mp_{n-1}+p_{n-2}}{Mq_{n-1}+q_{n-2}} \sim \frac{p_{n-1}}{q_{n-1}}\right);$$

or
$$\frac{1}{mq_{n-1}+q_{n-2}} < \frac{2}{Mq_{n-1}+q_{n-2}}, \text{ i.e. } M-2m < \frac{q_{n-2}}{q_{n-1}}.$$

Now
$$M = m_n + \frac{1}{m_{n+1}+}\,\frac{1}{m_{n+2}+}\,\frac{1}{m_{n+3}+}\cdots,$$

and
$$\frac{q_{n-2}}{q_{n-1}} = \frac{1}{m_{n-1}+}\,\frac{1}{m_{n-2}+}\,\frac{1}{m_{n-3}+}\cdots+\frac{1}{m_1}.*$$

Thus we have to take $m > \frac{1}{2}m_n$; *or* $= \frac{1}{2}m_n$ *(m_n being even), in the case where*

$$\frac{1}{m_{n+1}+}\,\frac{1}{m_{n+2}+}\,\frac{1}{m_{n+3}+}\cdots < \frac{1}{m_{n-1}+}\,\frac{1}{m_{n-2}+}\,\frac{1}{m_{n-3}}\cdots,$$

but not otherwise.

The special advantage of this form of condition is that it involves the m's only, and in such a way that a glance at their successive values shows at once how many "intermediate fractions" must be taken between each pair of convergents

---

* *i.e.* supposing (as throughout) $a < b$, and $\frac{1}{m_1}$ the first convergent. Then for $\frac{b}{a}$ we just get throughout the reciprocal fractions; the only difference being, that instead of the condition here for taking $m = \frac{1}{2}m_n$, we shall have

$$\frac{1}{m_{n+1}+}\,\frac{1}{m_{n+2}+}\,\frac{1}{m_{n+3}+}\cdots < \frac{1}{m_{n-1}+}\,\frac{1}{m_{n-2}+}\,\frac{1}{m_{n-3}+}\cdots+\frac{1}{m_2},$$

the right-hand fraction thus curtailed being the value of $\frac{p_{n-2}}{p_{n-1}}$. The practical result will, in general, be the same: the possibility of a difference is explained by remembering that the nearer of two fractions to another does not necessarily have its reciprocal nearer to that of the other. As m increases, the fraction $\frac{mp_{n-1}+p_{n-2}}{mq_{n-1}+q_{n-2}}$ continually approaches $\frac{p_{n-1}}{q_{n-1}}$, and therefore $\frac{a}{b}$, till it reaches $\frac{p_n}{q_n}$.

to give the complete series of fractions, *i.e.* that which includes every fraction nearer to the final value than any with smaller denominator.

These m's are, of course, the fundamental quantities in the formation of the convergents, and appear immediately as the quotients in the process for finding the G.C.M. of a and b. The (equally obvious) remainders denoted by the r's in the

$$\begin{array}{llllll} a) & b\ (m_1 & & & & \\ & \underline{m_1 a} & & & & \\ & r_1) & a\ (m_2 & & & \\ & & \underline{m_2 r_1} & & & \\ & & r_2) & r_1\ (m_3 & & \\ & & & \underline{m_3 r_2} & & \\ & & & r_3 & & \end{array}$$

subjoined scheme, have likewise a significance which is worthy of notice. It will be found that these r's give the successive values of $pb \sim qa$; and therefore, although they do not indicate the degree of approximation of the converging *fractions* apart from a knowledge of the q's, they measure at once the amount of divergence of the *multiples* of a and b, and are therefore for them of fundamental importance, showing even before the p's and q's are formed how far it will be necessary to go to obtain any required degree of approximation of these multiples. For proof of the fact it is sufficient to observe that we have

$$r_3 = -m_3 r_2 + r_1, \quad r_4 = -m_4 r_3 + r_2, \text{ \&c.};$$

so that $-r_3$, $+r_4$, $-r_5$, $+r_6$, &c. follow the same law of formation as the p's and the q's; also obviously (since $p_1 = 1$, $q_1 = m_1$, $p_2 = m_2$, $q_2 = m_1 m_2 + 1$), $r_1 = -m_1 a + b = p_1 b - q_1 a$, $r_2 = a - m_2 r_1 = q_2 a - p_2 b$. We therefore have universally $r_n = p_n b \sim q_n a$, the required result.

An important practical point remains, which affects more particularly the theory of the multiples. The numbers a and b, in practice, will not, in general, be themselves strictly accurate, but only approximate values of certain quantities A and B with which we may be concerned, and which we may suppose expressed by whole numbers in terms of any required decimal of the usual unit of measurement: if these numbers be correct to the last figure, there will still be an uncertainty of $\pm \frac{1}{2}$ in the values of A and B. This consideration will not be without its effect on the calculation of the series of fractions converging to the ratio of the two quantities,

as it will plainly not be worth while to proceed much further than a convergent whose difference from $\frac{a}{b}$ is less than the possible inaccuracy of the latter as a measure of $\frac{A}{B}$: still we shall even then be approaching what we may consider as the most probable value of the required ratio. If, however, it is approximate common multiples of A and B that are required, the case is quite otherwise, as beyond a certain point we shall be laying ourselves open to the risk of greater divergence the further we go in the series of "converging multiples" obtained for the numbers a and b, owing to the above uncertainty. If for instance, to take a well-known example, the cycles are required after which the sun, moon, and moon's node will return approximately to the same relative positions; taking a mean lunation $=29.5306$ days, and the mean synodic period of the earth and moon $= 346.6196$ days, such cycles will be obtained by forming the successive convergents to the ratio of the above numbers. But whereas, if these numbers were exact, we should get a continually closer approximation of the multiples until we reached the final result that 295306 synodic periods precisely equalled 3466196 lunations,* the possible error of half a ten-thousandth of a day in each of the given values, even if correct to the last decimal place, will entail a possible divergence between these two periods of about 188 days, or more than six months, so that there is absolutely no more assurance than the above number of synodic periods will be commensurate with an exact number of lunations than any other number whatever. The final result therefore is absolutely useless, and the question arises how far the intermediate approximate results will hold.

Let it be required to generally investigate how far we may go with advantage. The n^{th} remainder r_n, in the G.C.M. process, we have seen measures the divergence $p_n b \sim q_n a$: if a and b are each subject to an uncertainty of $\pm \frac{1}{2}$ as representatives of A and B, r_n will be subject to an uncertainty of $\pm \frac{1}{2} u_n = \pm \frac{1}{2} (p_n + q_n)$ in its representation of $p_n B \sim q_n A$, *i.e.* its range of uncertainty $u_n = p_n + q_n$, which increases as r_n diminishes. Now observe that

(1) So long as u_n is not greater than r_n, we get (almost necessarily) a continually closer approximation of the multiples $p_n B$, $q_n A$.

* Ignoring the existence of common factors of the two numbers. a and b may have a common factor, not removed, and the general results will hold throughout; only the final values of p and q will be then sub-multiples of a and b.

(2) Even until $u_n = r_{n-1}$, it will be seen from an inspection of the scheme for the G.C.M. process given above, that the remainder from the division which gives the quotient m_n will have a range of uncertainty not exceeding the divisor, so that the complete quotient M_n will be liable to an uncertainty not exceeding $\pm \frac{1}{2}$; and the value of m_n will not, in general, be affected (as we may suppose the error will not, as a rule, nearly = its maximum value), and cannot in any case be subject to a correction of more than unity. Up to the n^{th} pair of multiples, then, it will be worth while to go to get the probably best approximation, when $u_n = r_{n-1}$, although we cannot be quite sure that $p_n B$, $q_n A$ will be closer together than the preceding pair of multiples. Beyond such a point the values of the m's will be quite indeterminate; and all we can say is that the proper series of pairs of multipliers of A, B will be of the forms

$$kp_n + lp_{n-1}, \quad kq_n + lq_{n-1}, \quad (k > l),$$

where p_n, q_n is the last pair to be depended on.

So far the tests involve (through the u's) the p's and the q's, and can only be applied when these have been calculated; but $r_{n-1} = p_{n-1}b \sim q_{n-1}a$, and to this either of the expressions

$$b\left(p_{n-1} \sim q_{n-1}\frac{p_n}{q_n}\right), \quad a\left(p_{n-1}\frac{q_n}{p_n} \sim q_{n-1}\right), \text{ i.e. } \frac{b}{q_n} \text{ or } \frac{a}{p_n}$$

will be (roughly) equal; (each, however, differing from it in excess; for in the substitutions of $\frac{p_n}{q_n}$ and $\frac{q_n}{p_n}$, for $\frac{a}{b}$ and $\frac{b}{a}$ respectively, we necessarily in each case increase the larger term or diminish the smaller): we thus have roughly

$$r_{n-1} = \frac{a}{p_n} = \frac{b}{q_n} = \frac{a+b}{p_n+q_n} = \frac{a+b}{u_n},$$

though strictly somewhat less.* Approximately, therefore

(1) $u_n = r_n$ gives $r_{n-1} r_n = a + b = u_n u_{n+1}$,

(2) $u_n = r_{n-1}$ gives $r^2_{n-1} = a + b = u_n^2$.

---

* Otherwise thus: it is easily seen that (strictly) $a + b = r_{n-1}u_n + r_n u_{n-1}$; and the latter term will usually be small compared with the former, for we have

$$u_n = m_n u_{n-1} + u_{n-2}, \quad r_{n-1} = m_{n+1} r_n + r_{n+1};$$

so that the ratio of

$$r_{n-1}u_n \text{ to } r_n u_{n-1} = (m_n + \text{frn.})\,(m_{n+1} + \text{frn.}).$$

The rule given by this last, that we may proceed with the formation of the multiples so long as r_{n-1} is not greater than $\sqrt{(a+b)}$ is the simplest general result; which will be generally sufficient in practice.

When, as will usually be the case, there is no value of n for which one of the equalities exactly holds, we may perhaps proceed one step further than the rules would strictly allow, if we thus reach a stage where the value of r is only slightly less, or of u slightly greater, than it should be. It may be observed that as $a+b$, in the approximate criteria, is somewhat too large this will more particularly apply to the use of the rules involving the r's only, especially if $m_n m_{n+1}$ be inconsiderable. (See note).

In certain cases (*e.g.* where a and b have already had a common factor removed, or from a more accurate knowledge of the values of A and B), the uncertainty in the last places of a and b may be less than $\pm\frac{1}{2}$: in others it may be greater. If for any reason it $=\pm\frac{1}{2}z$ instead of $\pm\frac{1}{2}$ (where z is less or greater than 1), we shall have $u_n = z(p_n + q_n)$, The conditions $u_n < r_n$, $u_n < r_{n-1}$ will stand as before; but now we shall have approximately $r_n = \dfrac{z(a+b)}{u_{n+1}}$, giving (1) $r_{n-1}r_n$ not $< z(a+b)$, (2) r_{n-1} not $< \sqrt{\{z(a+b)\}}$.

As an illustration we may take the example referred to on page 84, and form the converging multiples of the numbers there given, on the supposition that they are correct to the last place. The complete G.C.M. process is:—

```
295306) 3466196 (11                      1140) 4486 (3
        295306                                 3420
        -------                                ----
         513136                                1066) 1140 (1
         295306                                      1066
        -------                                      ----
         217830) 295306 (1                             74) 1066 (14
                 217830                                     74
                 ------                                     ---
                  77476) 217830 (2                          326
                         154952                             296
                         ------                             ---
                          62878) 77476 (1                    30) 74 (2
                                 62878                           60
                                 -----                           --
                                 14598) 62878 (4                 14(30 (2
                                        58392                       28
                                        -----                       --
                                         4486) 14598 (3              2) 14 (7
                                               13458                    14
                                               -----                    --
                                                1140
```

We have:—

Sum of Nos. $= 3761502$.

Sq. Root (nearly) 1940.

5th Remainder $= 4486$.

6th " $= 1140$.

For the first seven convergents:—

m's	p's	q's	u's	r's	
11	1	11	12	217830	(1),
1	1	12	13	77476	(2),
2	3	35	38	62878	(3),
1	4	47	51	14598	(4),
4	19	223	242	4486	(5),
3	61	716	777	1140	(6),
(4	263	3087	3350	-74)	(7).

Seven convergents have been formed as r_6 is the first remainder $< \sqrt{(a+b)}$. It is, however, considerably less, so that the seventh is unlikely to be of much value; and this is confirmed when u_7 has been found. We have above in place of m_7 used the next higher integer 4, as being much nearer to the complete quotient: otherwise the divergence of the last pair of multiples (1066 in place of 1140) would show too little gain to set against the necessary increase from u_6 to u_7. (As the final value of u increases with that of m, it would not be worth while to take the higher number if only slightly nearer). Here, even the lower value of $u_7 = 2573$, while $r_6 = 1140$; so that the last result must be altogether set aside, and the best conclusion we can draw from the given data is that 716 lunations $= 61$ synodic periods within .1140 of a day, except for an uncertainty of $\pm \frac{1}{2}u = \pm .0388$.

Lower numbers will better illustrate the other points: *e.g.* let $\frac{a}{b} = \frac{387}{1768}$. The G.C.M. work shows the m's (and r's) to be

$$5\,(81) \quad 4\,(13) \quad 6\,(3) \quad 4\,(1) \quad 3\,(0).$$

The additional m's to give the complete series of fractions will be

$$2, 3 \quad 3, 4, 5 \qquad 3 \qquad 2$$

for they include $\frac{1}{2}m_2$, because 6 which follows it in the series of m's is greater than 5 which precedes; $\frac{1}{2}m_3$, because, the adjacent numbers being equal, the next preceding, 5, is greater than the next following, 3; but not $\frac{1}{2}m_4$, because the following number, 3, is the less: m_5 being odd, the question does not arise for it; otherwise its half would be available; (as we may consider $m_6 = \infty$).

```
337) 1766 (5
     1685
     ----
       81) 337 (4
           324
           ---
            13) 81 (6
                78
                --
                 3) 13 (4
                    12
                    --
                     1) 3 (3
                        3
                        -
```

The complete series of fractions, and their differences from $\frac{a}{b}$ therefore are

$$\text{Frns.}\quad \frac{1^*}{5},\ \frac{2}{11},\ \frac{3}{16},\ \frac{4^*}{21},\ \frac{13}{68},\ \frac{17}{89},\ \frac{21}{110},$$

$$\frac{25^*}{131},\ \frac{79}{414},\ \frac{104^*}{545},\ \frac{233}{1221},\ \frac{337^*}{1766}.$$

$$\text{Diffs.}\quad \frac{1}{1766}\times\frac{81}{5},\ \frac{175}{11},\ \frac{94}{16},\ \frac{13}{21},\ \frac{42}{68},\ \frac{29}{89},$$

$$\frac{16}{110},\ \frac{3}{131},\ \frac{4}{414},\ \frac{1}{545},\ \frac{1}{1221},\ 0.$$

It will be seen that the numerators of the intermediate fraction differences are distinguished by their relatively higher values; their law of formation from the r's (by addition of preceding r) should be noticed. Had we used $\frac{1}{2}m_4$, we should have obtained the fraction $\frac{54}{283}$, difference $\frac{7}{283b}$. This somewhat exceeds $\frac{3}{131b}$, the error of $\frac{25}{131}$.

Durham.

EXPANSION OF ELLIPTIC INTEGRALS BY ZONAL HARMONICS, WITH SOME DERIVED INTEGRALS AND SERIES.

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THE expansion was suggested by a comparison of two expressions for the annual supply of heat or light at any point of the earth's surface, due to solar radiation through a non-absorbing atmosphere.* On the one hand this is given by complete elliptic integrals of the three kinds, on the other by a doubly-zonal-harmonic series $\Sigma C_n P_{2n}(\sin\lambda) P_{2n}(\cos\epsilon)$, where λ denotes the latitude of the place, ϵ the obliquity of the ecliptic. The case $\lambda = 0$ leads to $E(\sin\epsilon) = \Sigma C_n' P_{2n}(\cos\epsilon)$, and in an identical manner $\epsilon = 90°$, leads to

$$E(\cos\lambda) = \Sigma C_n' P_{2n}(\sin\lambda);$$

in either case the discontinuity, which occurs in crossing the Arctic circle, when both variables are present, disappears. The complete elliptic integral $E(\sin\theta)$ thus receives a double interpretation, viz. it expresses the manner in which the annual heat-supply at the equator of a planet depends on the obliquity of the ecliptic (θ), or in the case of a planet whose axis lies in the plane of the ecliptic, it expresses the distribution of heat-supply in any latitude $90° - \theta$. The analogy with trigonometrical functions appears in the corresponding statements for the opposite extreme cases, viz. the dependenee of the heat-supply at the pole on the obliquity of the ecliptic θ, or the distribution in any latitude $90° - \theta$ for no obliquity, is expressed by $\sin\theta$.

The expansion is here obtained by direct analysis, it leads by an induction process to a simple evaluation of integrals of the type $\int_0^1 \{E(k) \text{ or } K(k)\} k'^{2n} dk'$, and a comparison with another method gives a finite and an infinite series for $\left(\frac{1.3\ldots(2n-1)}{2.4\ldots 2n}\right)^2$, which, when n is very great, merge into Wallis's well-known result for this limit. Incidentally the expansion of $(1-\mu^2)^{\frac{1}{2}(2p+1)}$ in zonal harmonics is given, which is an extension of a formula in Todhunter's *Laplace's Functions*, p. 115.

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* The reference is to the author's paper on "Distribution of Solar Radiation on the Surface of the Earth" in *Cambridge Philosophical Transactions*, Vol. XVI, Part 1.

§ 1. Starting from the well-known expansion

$$\sqrt{(1-\mu^2)} = \tfrac{1}{4}\pi - \tfrac{1}{2}\pi\,\Sigma_{n=1}^{n=\infty} \frac{(4n+1)\,a_n^{\,2}P_{2n}(\mu)}{(2n-1)(2n+2)},$$

where a_n stands for $\dfrac{1.3\ldots(2n-1)}{2.4\ldots2n}$, write

$$\mu = \sin\theta\sin\phi = \cos\theta\cos\tfrac{1}{2}\pi + \sin\theta\sin\tfrac{1}{2}\pi\cos(\tfrac{1}{2}\pi - \phi)$$

and then expand each zonal harmonic. For this purpose we have

$$P_{2n}(\mu) = 2\Sigma_{s=0}^{s=n} \frac{(-1)^s \lfloor n-2s}{2^{2n}\lfloor n+s \lfloor n-s} \sin^{2s}\theta\,P_{2n}^{\,2s}(\cos\theta)\cos 2s\phi,$$

in which $P_{2n}^{\,2s}(\mu) = \dfrac{d^{2s}}{d\mu^{2s}}P_{2n}(\mu)$; the factor 2 is to be omitted in the opening term. The substitution gives

$$\sqrt{(1-\sin^2\theta\sin^2\phi)} = \tfrac{1}{4}\pi + \tfrac{1}{2}\pi\,\Sigma_{n=1}^{s=\infty} \frac{(-1)^{n+1}(4n+1)\,a_n^{\,2}}{(2n-1)(2n+2)} P_{2n}(\cos\theta)$$

$$+ \pi\,\Sigma_{s=1}^{s=\infty}\cos 2s\phi\,\Sigma_{n=s}^{n=\infty} \frac{(-1)^{n+1}(4n+1)\lfloor 2n-2s\,.\,a_n^{\,2}}{2^{2n}(2n-1)(2n+2)\lfloor n+s\lfloor n-s}$$

$$\times \sin^{2s}\theta P_{2n}^{\,2s}(\cos\theta)\ldots(1).$$

Integrate both sides with regard to ϕ from 0 to $\frac{1}{2}\pi$, and we obtain

$$\int_0^{\frac{1}{2}\pi}\sqrt{(1-\sin^2\theta\sin^2\phi\,d\phi)}$$

$$= \tfrac{1}{8}\pi^2 + \tfrac{1}{4}\pi^2\,\Sigma_1^{\infty} \frac{(-1)^{n+1}(4n+1)\,a_n^{\,2}}{(2n-1)(2n+2)} P_{2n}(\cos\theta);$$

or, as we may more conveniently write it, with k for $\sin\theta$ and k' for $\cos\theta$,

$$E(k) = \tfrac{1}{8}\pi^2 + \tfrac{1}{4}\pi^2\,\Sigma_1^{\infty} \frac{(-1)^{n+1}\,4n+1)\,a_n^{\,2}}{(2n-1)(2n+2)} P_{2n}(k')\ \ldots(2).$$

Again, if we multiply by $\cos 2s\phi$ before integrating, we have

$$E_{2s}(k) = \tfrac{1}{4}\pi^2\,\Sigma_{n=s}^{n=\infty} \frac{(-1)^{n+1}(4n+1)\lfloor 2x-2s\,.\,a_n^{\,2}}{2^{2n}(2n-1)(2n+2)\lfloor n-s\lfloor n+s} k^{2s}P_{2n}^{\,2s}(k')$$

$$\ldots\ldots\ldots(3),$$

where $E_{2s}(k)=\int_0^{\frac{1}{2}\pi}\cos 2s\phi\,\sqrt{(1-k^2\sin^2\phi)}\,d\phi$. Similar work, starting from the expansion $(1-\mu^2)^{-\frac{1}{2}}$, gives

$$\left.\begin{aligned} K(k) &= \tfrac{1}{4}\pi^2+\tfrac{1}{4}\pi^2\,\Sigma_1^\infty\,(-1)^n\,(4n+1)\,a_n^{\,2}P_{2n}(k') \\ K_{2s}(k) &= \tfrac{1}{4}\pi^2\,\Sigma_{n=s}^{n=\infty}\frac{(-1)^n\,(4n+1)\,a_n^{\,2}\,\lfloor 2n-2s}{2^{2n}\,\lfloor n-s\,\lfloor n+s}\,k^{2s}P_{2n}^{\,2s}(k')\end{aligned}\right\}$$
$$\ldots\ldots(4).$$

Or, again, if we integrate from 0 to ϕ, we obtain

$$E(k,\phi)=\frac{2\phi E}{\pi}+\pi\,\Sigma_{s=1}^{s=\infty}\,\frac{\sin 2s\phi}{s}$$
$$\times\,\Sigma_{n=s}^{n=\infty}\,\frac{(-1)^{n+1}\,(4n+1)\,\lfloor 2n-2s\,a_n^{\,2}}{2^{2n+1}\,(2n-1)\,(2n+2)\,\lfloor n-s\,\lfloor n+s}\,k^{2s}P_{2n}^{\,2s}(k')\,;$$

$$K(k,\phi)=\frac{2\phi K}{\pi}+\pi\,\Sigma_{s=1}^{s=\infty}\,\frac{\sin 2s\phi}{s}$$
$$\times\,\Sigma_{n=s}^{n=\infty}\,\frac{(-1)^n\,(4n+1)\,\lfloor 2n-2s\,a_n^{\,2}}{2^{2n+1}\lfloor n-s\,\lfloor n+s}\,k^{2s}P_{2n}^{\,2s}(k').$$

Immediately from (2) and the first series of (4) follows the system of integrals

$$\left.\begin{aligned} &\int_0^1 E(k)\,dk'=\tfrac{1}{8}\pi^2 \\ &\int_0^1 E(k)\,P_{2n}(k')\,dk'=\tfrac{1}{4}\pi^2\,(-1)^{n+1}\,\frac{a_n^{\,2}}{(2n-1)\,(2n+2)} \\ &\int_0^1 K(k)\,dk'=\tfrac{1}{4}\pi^2 \\ &\int_0^1 K(k)\,P_{2n}(k')\,dk'=\tfrac{1}{4}\pi^2\,(-1)^n\,a_n^{\,2}\end{aligned}\right\}\ldots(5).$$

§ 2. From these I propose to deduce

$$\int_0^1 E(k)\,.\,k'^{2s}dk'=\tfrac{1}{4}\pi^{2s}a_s a_{s+1}\text{ and }\int_0^1 K(k)\,.\,k'^{2s}dk'=\tfrac{1}{4}\pi^2 a_s^{\,2}\ldots(6),$$

the formulæ being suggested by using the values for $P_2(k')$, $P_4(k')$, ... in succession. To establish these, write

$$P_{2n}=\Sigma\,(-1)^{n-s}A_{2s}x^{2s}=\Sigma\,(-1)^s\,B_{2s}\,(1-x^2)^s,$$

in which

$$A_{2s} = {}_nC_s \cdot \frac{(2s+1).(2s+3)\dots(2n+2s-1)}{2.4\dots 2n}$$

$$= {}_nC_s \cdot \frac{1.3\dots(2n+2s-1)}{1.3\dots(2s-1).2.4\dots 2n},$$

$$B_{2s} = {}_nC_s \cdot \frac{(2n+1).(2n+3)\dots(2n+2s-1)}{2.4\dots 2s}$$

$$= {}_nC_s \cdot \frac{1.3\dots(2n+2s-1)}{1.3\dots(2n-1).2.4\dots 2s},$$

with ${}_nC_s = s\dfrac{\lfloor n}{\lfloor s \lfloor n-s}$, and we infer the relation

$$A_{2s} \times a_s = B_{2s} \times a_n \quad \dots\dots\dots\dots\dots\dots\dots(7).$$

Now

$$\pi a_n^2 = \int_0^\pi P_{2n}(\cos\theta)\,d\theta = \int_0^\pi \Sigma(-1)^s B_{2s} \sin^{2s}\theta\,d\theta = \pi\Sigma(-1)^s B_{2s} a_s,$$

therefore

$$\tfrac{1}{4}\pi^2(-1)^n a_n^2 = \tfrac{1}{4}\pi^2\,\Sigma(-1)^{n-s} B_{2s} a_n a_s = \tfrac{1}{4}\pi^2\,\Sigma(-1)^{n-s} A_{2s} a_s^2 \dots(8).$$

The left-hand member of this equation is the value of $\int_0^1 K(k)\,P_{2n}(k')\,dk'$ given in (5), and the right-hand member is the value of the separate terms of the same integral, if the formula suggested in (6) is true.

It is clear then that the formula follows by an induction, viz. if it is true up to k'^6, for instance, the application of (8) for $P_8(k')$ establishes the new formula for k'^8. The same mode of proof applies to E, and, indeed, generally to

$${}_pE = \int_0^{\frac{1}{2}\pi} (1 - k^2\sin^2\theta)^{\frac{1}{2}(2p+1)}\,d\theta.$$

For this we require the expansion (true to $p = -1$),

$$\left.\begin{aligned}(1-\mu^2)^{\frac{1}{2}(2p+1)} &= \tfrac{1}{2}\pi a_{p+1} + \tfrac{1}{2}\pi(-1)^{p+1} \\ &\times \Sigma_{n=1}^{n=\infty} \frac{(4n+1)\,a_n^2\,\{1.3\dots(2p+1)\}^2\,P_{2n}(\mu)}{(2n+2).(2n+4)\dots(2n+2p+2).(2n-1).(2n-3)\dots(2n-2p-1)} \\ \text{say } (1-\mu^2)^{\frac{1}{2}(2p+1)} &= {}_pD_0 + \Sigma_p D_{2n} P_{2n}\end{aligned}\right\} \dots\dots\dots(9).$$

To prove this use

$$\frac{d}{d\mu}(1-\mu^2)\frac{d}{d\mu}(1-\mu^2)^{\frac{1}{2}(2p+1)}$$

$$=-(2p+1)(2p+2)(1-\mu^2)^{\frac{1}{2}(2p+1)}+(2p+1)^2(1-\mu^2)^{\frac{1}{2}(2p-1)},$$

which gives ${}_pD_{2n}(2n-2p-1)(2n+2p+2)=-(2p+1)^2{}_{p-1}D_{2n}$ to connect consecutive types, and from the known formula for $(1-\mu^2)^{\frac{1}{2}}$ leads to the more general form in (9). In (9) write $\mu=\sin\theta\sin\phi$,

$$(1-\sin^2\theta\sin^2\phi)^{\frac{1}{2}(2p+1)}=D_0+\Sigma_{n=1}^{n=\infty}{}_pD_{2n}(-1)^n a_n P_{2n}(\cos\theta)$$

$$+2\Sigma_{s=1}^{s=\infty}\cos 2s\phi\,\Sigma_{n=s}^{n=\infty}\frac{{}_pD_{2n}(-1)^n\lfloor 2n-2s}{2^{2n}\lfloor n-s\lfloor n+s}\sin^{2s}\theta P_{2n}^{2s}(\cos\theta)$$

from which

$${}_pE(k)=\tfrac{1}{2}\pi D_0+\tfrac{1}{2}\pi\,\Sigma_{n=p}^{n=\infty}{}_pD_{2n}(-1)^n a_n P_{2n}(k')\ldots\ldots(10),$$

a generalisation of (2), and immediately

$$\int_0^1 {}_pE(k)P_{2n}(k')\,dk'=\frac{\pi(-1)^n}{2(4n+1)}a_n\,{}_pD_{2n}\ \ldots\ldots.(11)$$

which corresponds to (5).

Now, from (9) follows

$$\frac{2\,{}_pD_{2n}}{4n+1}=\int_{-1}^{+1}(1-\mu^2)^{\frac{1}{2}(2p+1)}P_{2n}(\mu)\,d\mu=\int_0^\pi\sin^{2p+2}\theta P_{2n}(\cos\theta)\,d\theta$$

$$=\int_0^\pi\Sigma(-1)^s B_{2s}\sin^{2s+2p+2}\theta\,d\theta=\pi\Sigma(-1)^s B_{2s}a_{s+p+1}.$$

Therefore

$$\int_0^1{}_pE(k)P_{2n}(k')\,dk'=\frac{\pi(-1)^n}{2(4n+1)}a_n\,{}_pD_{2n}=\tfrac{1}{4}\pi^2\,\Sigma(-1)^{n-s}B_{2s}a_n a_{s+p+1}$$

$$=\tfrac{1}{4}\pi^2\,\Sigma(-1)^{n-s}A_{2s}a_s a_{s+p+1}.$$

The inductive process used above now proves

$$\int_0^{+1}{}_pE(k)k'^{2s}dk'=\tfrac{1}{4}\pi^2\,a_s a_{s+p+1}\ \ldots\ldots\ldots\ldots(12).$$

When $p=0$, this gives the formula for E, when $p=-1$ that for K.

The process used here and in proving (8) gives a curious result which may be stated as follows:—If in the series for $P_{2n}(x)$, x^{2s} is replaced by $a_s a_{s+p+1}$, where $a_s = \dfrac{1.3\,..(2s-1)}{2.4...2s}$, the summation takes the form

$$\frac{(-1)^{n+p+1} a_n^{\,2} \{1.3...(2p+1)\}^2}{(2n+2)(2n+4)...(2n+2p+2)(2n-1)(2n-3)...(2n-2p-1)};$$

p may be -1, 0, or any positive integer. With $p=-1$, x^{2s} is replaced by $a_s^{\,2}$ and the summation yields $(-1)^n a_n^{\,2}$.

§ 3. A large number of series may be summed by using particular values in the above formulæ, a few of which we indicate. In the original formula for $\sqrt{(1-\mu^2)}$, write $\mu=0$, or in the formula for E, $k=0$, $k'=1$, then

$$\frac{2}{\pi} = \tfrac{1}{2} + \Sigma_1^\infty \frac{(-1)^{n+1}(4n+1)\,a_n^{\,3}}{(2n-1)(2n+2)} \quad ....(13),$$

and writing $k=1$, $k'=0$ in the formula for E,

$$\frac{4}{\pi^2} = \tfrac{1}{2} - \Sigma_1^\infty \frac{(4n+1)\,a_n^{\,4}}{(2n-1)(2n+2)} \quad ....(14).$$

The formula for $(1-\mu^2)^{-\frac{1}{2}}$ with $\mu=0$, or that for K with $k=0$, $k'=1$ gives

$$\frac{2}{\pi} = 1 + \Sigma(-1)^n(4n+1)\,a_n^{\,3} \quad ....(15)$$

of slower convergence than (13).

Again, it is possible to obtain the value of $\int_0^1 E(k)\,k'^{2m}dk'$ in a finite series by making use of

$$\int_0^1 P_{2n}(k').k'^{2m}dk' = \frac{2m(2m-2)...(2m-2n+2)}{(2m+2n+1)(2m+2n-1)...(2m+1)}$$

in each term of the expansion of E. Comparing the result with (6), we have

$$a_m a_{m+1} = \frac{1}{2(2m+1)} + \Sigma_{n=1}^{n=m} \frac{(-1)^{n+1}(4n+1)\,a_n^{\,3}}{(2n-1)(2n+2)}$$
$$\times \frac{2m.(2m-2)...(2m-2n+2)}{(2m+2n+1).(2m+2n-1)...(2m+1)} \quad ...(16).$$

Suppose m to be very large in this result, and we get in the limit

$$a_m a_{m+1} = \frac{1}{(2m+1)}\left[\tfrac{1}{2} + \Sigma \frac{(-1)^{n+1}(4n+1)\,a_n^2}{(2n-1)(2n+2)}\right];$$

or, quoting (13), this is $a_m a_{m+1} = \dfrac{2}{(2m+1)\,\pi}$, or the limit of a_m^2 when m is indefinitely increased is $\dfrac{1}{m\pi}$, which is Wallis's well-known result.

The K formula, treated in the same way, gives

$$a_m^2 = \frac{1}{2m+1} + \Sigma_{n=1}^{n=m} (-1)^{n+1}(4n+1)\,a_n^2$$

$$\times \frac{2m.(2m-2)\ldots(2m-2n+2)}{(2m+2n+1).(2m+2n-1)\ldots(2m+1)} \ldots(17),$$

leading to the limit $a_m^2 = \dfrac{1}{2m+1}\left[1 + \Sigma\,(-1)^n(4n+1)\,a_n^2\right]$, giving the same value as before when (15) is used.

The general formula for ${}_pE$, treated in this way, yields

$$a_m a_{m+p+1} = \frac{a_{p+1}}{2m+1}$$

$$+ \Sigma_{n=1}^{n=m} \frac{(-1)^{n+p+1}(4n+1)\,a_n^2.\{1.3\ldots(2p+1)\}^2}{(2n+2).(2n+4)\ldots(2n+2p+2).(2n-1).(2n-3)\ldots(2n-2p-1)}$$

$$\times \frac{2m.(2m-2)\ldots(2m-2n+2)}{(2m+2n+1)(2m+2n-1)\ldots(2m+1)} \ldots(18),$$

of which (16) and (17) are the particular cases, when $p=0$, $p=-1$.

A rather different type is obtained by using for the same comparison the ordinary series $K = \tfrac{1}{2}\pi\,(1 + \Sigma_1^{\infty} a_n^2 k^{2n})$.

Then

$$\tfrac{1}{2}\pi a_m^2 = \int_0^{\prime} K(k)\,k'^{2m}\,dk' = \tfrac{1}{2}\pi \int_0^1 \Sigma a_n^2 k^{2n} k'^{2m}\,dk'$$

$$= \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} \Sigma a_n^2 \sin^{2n+1}\theta \cos^{2m}\theta\,d\theta$$

$$= \tfrac{1}{2}\pi\,\Sigma \frac{1.3\ldots(2n-1).a_n}{(2m+1).(2m+3)\ldots(2m+2n+1)},$$

or

$$a_m^2 = \frac{2}{\pi(2m+1)}\left[1 + \Sigma_{n=1}^{n=\infty} \frac{1.3\ldots(2n-1).a_n}{(2m+3).(2m+5)\ldots(2m+2n+1)}\right]$$
......(19).

With ${}_pE$ the method leads to

$$a_m a_{m+p+1} = \frac{2}{\pi(2m+1)}$$
$$\times\left[1 + \Sigma_{n=1}^{n=\infty} \frac{(2p+1)(2p-1)\ldots(2p-2n+3).(-1)^n a_n}{(2m+3)(2m+5)\ldots(2m+2n+1)}\right]$$
......(20).

These are infinite series, which are well adapted for giving an approximation to a_m^2 or $a_m a_{m+p+1}$ for high values of m, as m appears only in the denominators. For low values of m the convergence is slow; for example, on writing $m=0$, we have

$$\tfrac{1}{4}\pi = 1 - \Sigma_{n=1}^{n=\infty} \frac{a_n}{4n^2-1},$$

and

$$\tfrac{1}{2}\pi = 1 + \Sigma_{n=1}^{n=\infty} \frac{a_n}{2n+1}.$$

The latter of these may be got by writing the expansion of $\frac{1}{\sin\theta}\log\frac{1+\sin\theta}{1-\sin\theta}$ and integrating from 0 to $\frac{1}{2}\pi$; for the former the additional factor $\cos^2\theta$ appears; and it is probable that other results can be obtained independently of the elliptic integral expansions.

Another example of the usual series for K is

$$\int_0^1 Kk'^{2p+1}dk' = \int_0^{\frac{1}{2}\pi} K\sin^{2p+2}\theta\, d\theta = \tfrac{1}{2}\pi\int_0^{\frac{1}{2}\pi} \Sigma a_n^2 \sin^{2n+2p+2}\theta\, d\theta$$

$$= \tfrac{1}{4}\pi^2\left[\tfrac{1}{2} + \Sigma a_n^2 a_{n+p+1}\right] = \int_c^1 {}_p\frac{E.dk'}{k}$$

by (6), and the expansion $k^{-1} = (1-k'^2)^{-\frac{1}{2}} = 1 + \Sigma a_n k'^{2n}$. With $p=0$, this is

$$\int_0^1 Kk\,dk' = \int_0^1 \frac{Edk'}{k}\ldots\ldots\ldots\ldots\ldots\ldots(21).$$

CONTENTS.

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But a much better convergence is obtained by using the zonal expansion; thus

$$\int_0^1 \frac{E}{k}\,dk' = \tfrac{1}{2}\pi \int_0^1 E\,dk' \left[1 + \Sigma\,(4n+1)\,a_n^2 P_{2n}(k')\right]$$

$$= \tfrac{1}{8}\pi^3 \left[\tfrac{1}{2} + \Sigma \frac{(-1)^{n+1}(4n+1)\,a_n^3}{(2n-1)(2n+2)}\right] \dots (22).$$

I have developed to some extent a corresponding series of results for $E_{2,1}$, but the complexity seriously detracts from any interest they may possess. The work turns on the expansion $(1-\mu^2)^{\frac{1}{2}n}$ for n odd or even in a series of associated functions which can be effected without much trouble.

§ 4. In conclusion it may be observed that the original expansion for E may be got by transforming

$$(1-k^2)\frac{d^2E}{dk^2} + \frac{1-k^2}{k}\frac{dE}{dk} + E = 0$$

to k' as variable; this leads to

$$(1-k'^2)\frac{d^2E}{dk'^2} - 2k'\frac{dE}{dk'} + E - \frac{1-k'^2}{k'}\frac{dE}{dk'} = 0.$$

A term $A_{2n}P_{2n}(k')$ in the expansion of E substituted in this equation gives

$$\{1 - 2n(2n+1)\}\,A_{2n}P_{2n} - A_{2n}(1-k'^2)\frac{1}{k'}\frac{dP_{2n}}{dk'},$$

or
$$\frac{1}{k'}\left[(4n^2-1)\,A_{2n}k'P_{2n} + 2nA_nP_{2n-1}\right].$$

Apply $(4n+1)\,k'P_{2n} = (2n+1)\,P_{2n+1} + 2nP_{2n-1}$ and equate to zero the coefficient of P_{2n-1}, and we get

$$\frac{A_{2n}}{A_{2n-2}} = -\frac{(4n+1)\,.\,(2n-1)^2\,.\,(2n-3)}{(4n-3)\,.\,(2n)^2\,.\,(2n+2)},$$

the correct relation.

This leaves A_0 undetermined, but

$$A_0 = \int_0^{\frac{1}{2}\pi} E\sin\theta\,d\theta = \int_0^{\frac{1}{2}\pi}\sin\theta\,d\theta\int_0^{\frac{1}{2}\pi}\sqrt{(1-\sin^2\theta\sin^2\phi)}\,d\phi$$

$$= \int_0^{\frac{1}{2}\pi}\sin\phi\,d\phi\int_0^{\frac{1}{2}\pi}\sin\theta\sqrt{(\cos^2\theta + \cot^2\phi}\,d\theta$$

$$= \int_0^{\frac{1}{2}\pi}\sin\phi\,d\phi\int_0^1\sqrt{(x^2+\cot^2\phi)}\,dx.$$

But $\int_0^1 dx\sqrt{(\cot^2\phi + x^2)} = \frac{1}{2\sin\phi} + \frac{1}{4}\cot^2\phi \log\frac{1+\sin\phi}{1-\sin\phi}$;

therefore

$$4A_0 - \pi = \int_0^{\frac{1}{2}\pi} \frac{\cos^2\phi}{\sin\phi}\log\frac{1+\sin\phi}{1-\sin\phi}\,d\phi = \int_0^{\frac{1}{2}\pi}\frac{1}{\sin\phi}\log\frac{1+\sin\phi}{1-\sin\phi}\,d\phi - \pi,$$

integrating by parts, or

$$A_0 = \frac{1}{4}\int_0^{\frac{1}{2}\pi}\frac{1}{\sin\phi}\log\frac{1+\sin\phi}{1-\sin\phi}\,d\phi = \frac{1}{8}\pi^2.$$

For K, the opening constant reduces to

$$\frac{1}{3}\int_0^{\frac{1}{2}\pi}\frac{1}{\sin\phi}\log\frac{1+\sin\phi}{1-\sin\phi}\,d\phi \text{ or } \frac{1}{6}\pi^2.$$

As regards K, the series becomes infinite for $k' = 1$ as it should do, and is finite for all other values. The various integrals used, $\int_0^1 K dk'$, $\int_0^1 K k'^{m} dk'$, are finite, though containing an infinite element, just as $\int_0^1 \frac{1}{\sqrt{(1-\mu^2)}}$ is finite.

TABLE OF BESSEL'S FUNCTIONS Y_0 AND Y_1.

By *B. A. Smith*, Melbourne University.

THIS table has been calculated from a formula given in Messrs. Gray and Matthews' *Treatise on Bessel Functions*, viz.

$$Y_0(x) = J_0(x)\log x + \frac{x^2}{2^2} - (1+\tfrac{1}{2})\frac{x^4}{2^2.4^2} + \dots .$$

The Y_1 functions have been calculated from the series obtained by differentiation of the above.

The values obtained have been checked by differences, and are subject to errors of not more than 2 in the last decimal place.

The following formulæ may be used for interpolation,

$$Y_0(x+h) = Y_0(x)\times\{1 - \tfrac{1}{2}h^2\} + Y_0'(x)\times\left\{h - \frac{h^2}{2x}\right\},$$

$$Y_0'(x+h) = Y_0'(x)\times\left\{1 - \frac{h}{x} - \tfrac{1}{2}h^2\left(1 - \frac{2}{x^2}\right)\right\} - Y_0(x)\times\left\{h - \frac{h^2}{2x}\right\}.$$

x	$Y_0(x)$	$Y_0'(x) = -Y_1(x)$	x	$Y_0(x)$	$Y_0'(x) = -Y_1(x)$
0	$-\infty$	$+\infty$	·32	− 1·0851	+ 3·3824
·01	− 4·6050	+ 100·0255	·33	− 1·0517	+ 3·2905
·02	− 3·9115	+ 50·0441	·34	− 1·0193	+ 3·2039
·03	− 3·5056	+ 33·3934	·35	− ·9876	+ 3·1223
·04	− 3·2172	+ 25·0744	·36	− ·9568	+ 3·0451
·05	− 2·9932	+ 20·0874	·37	− ·9267	+ 2·9721
·06	− 2·8100	+ 16·7660	·38	− ·8974	+ 2·9029
·07	− 2·6547	+ 14·3961	·39	− ·8687	+ 2·8371
·08	− 2·5201	+ 12·6209	·40	− ·8406	+ 2·7746
·09	− 2·4011	+ 11·2418	·41	− ·8132	+ 2·7151
·10	− 2·2943	+ 10·1399	·42	− ·7863	+ 2·6584
·11	− 2·1977	+ 9·2395	·43	− ·7600	+ 2·6042
·12	− 2·1090	+ 8·4902	·44	− ·7342	+ 2·5524
·13	− 2·0274	+ 7·8570	·45	− ·7089	+ 2·5028
·14	− 1·9516	+ 7·3149	·46	− ·6841	+ 2·4553
·15	− 1·8808	+ 6·8458	·47	− ·6598	+ 2·4097
·16	− 1·8145	+ 6·4358	·48	− ·6359	+ 2·3659
·17	− 1·7520	+ 6·0745	·49	− ·6125	+ 2·3238
·18	− 1·6929	+ 5·7538	·50	− ·5895	+ 2·2833
·19	− 1·6368	+ 5·4670	·51	− ·5668	+ 2·2443
·20	− 1·5834	+ 5·2094	·52	− ·5446	+ 2·2066
·21	− 1·5325	+ 4·9767	·53	− ·5227	+ 2·1702
·22	− 1·4838	+ 4·7652	·54	− ·5012	+ 2·1351
·23	− 1·4371	+ 4·5723	·55	− ·4800	+ 2·1012
·24	− 1·3928	+ 4·3956	·56	− ·4591	+ 2·0683
·25	− 1·3492	+ 4·2332	57	− ·4386	+ 2·0364
·26	− 1·3076	+ 4·0834	·58	− ·4184	+ 2·0055
·27	− 1·2675	+ 3·9448	·59	− ·3985	+ 1·9756
·28	− 1·2287	+ 3·8162	·60	− ·3789	+ 1·9466
·29	− 1·1911	+ 3·6965	·61	− ·3596	+ 1·9184
·30	− 1·1547	+ 3·5848	·62	− ·3405	+ 1·8909
·31	− 1·1194	+ 3·4804	·63	− ·3217	+ 1·8641

x	$Y_0(x)$	$Y_0'(x) = -Y_1(x)$	x	$Y_0(x)$	$Y_0'(x) = -Y_1(x)$
·64	− ·3032	+ 1·8380	·97	+ ·1914	+ 1·2189
·65	− ·2850	+ 1·8126	·98	+ ·2035	+ 1·2045
·66	− ·2670	+ 1·7879	·99	+ ·2155	+ 1·1902
·67	− ·2492	+ 1·7638	1·00	+ ·2273	+ 1·1761
·68	− ·2317	+ 1·7403			
·69	− ·2144	+ 1·7173	1.1	+ ·3382	+ 1·0420
·70	− ·1973	+ 1·6948	1·2	+ ·4361	+ ·9179
·71	− ·1805	+ 1·6728	1·3	+ ·5220	+ ·8011
·72	− ·1639	+ 1·6513	1·4	+ ·5965	+ ·6898
·73	− ·1475	+ 1·6303	1·5	+ ·6601	+ ·5830
·74	− ·1313	+ 1·6096	1·6	+ ·7132	+ ·4799
·75	− ·1153	+ 1·5894	1·7	+ ·7562	+ ·3803
·76	− ·0995	+ 1·5695	1·8	+ ·7894	+ ·2839
·77	− ·0839	+ 1·5500	1·9	+ ·8131	+ ·1909
·78	− ·0685	+ 1·5309	2·0	+ ·8277	+ ·1013
·79	− ·0532	+ 1·5121	2·1	+ ·8335	+ ·0153
·80	− ·0382	+ 1·4937	2·2	+ ·8308	− ·0668
·81	− ·0234	+ 1·4756	2·3	+ ·8202	− ·1447
·82	− ·0087	+ 1·4578	2·4	+ ·8020	− ·2182
·83	+ ·0058	+ 1·4402	2·5	+ ·7768	− ·2869
·84	+ ·0201	+ 1·4229	2·6	+ ·7448	− ·3505
·85	+ ·0342	+ 1·4059	2.7	+ ·7069	− ·4088
·86	+ ·0482	+ 1·3892	2·8	+ ·6633	− ·4614
·87	+ .0620	+ 1·3727	2·9	+ ·6147	− ·5084
·88	+ ·0757	+ 1·3564	3·0	+ ·5618	− ·5493
·89	+ ·0891	+ 1·3403	3·1	+ ·5051	− ·5841
·90	+ ·1025	+ 1·3244	3·2	+ ·4452	− ·6126
·91	+ ·1156	+ 1·3088	3·3	+ ·3828	− ·6348
·92	+ ·1286	+ 1·2934	3·4	+ ·3184	− ·6507
·93	+ ·1415	+ 1·2781	3·5	+ ·2529	− ·6603
·94	+ ·1542	+ 1·2630	3·6	+ ·1866	− ·6636
·95	+ ·1668	+ 1·2481	3·7	+ ·1203	− ·6608
·96	+ ·1792	+ 1·2334	3·8	+ ·0547	− ·6520

x	$Y_0(x)$	$Y_0'(x) = -Y_1(x)$	x	$Y_0(x)$	$Y_0'(x) = -Y_1(x)$
3·9	− ·0098	− ·6374	7·1	+ ·0412	+ ·4675
4·0	− ·0727	− ·6174	7·2	+ ·0873	+ ·4546
4·1	− ·1332	− ·5922	7·3	+ 1320	+ ·4375
4·2	− ·1909	− ·5620	7·4	+ ·1747	+ ·4168
4·3	− ·2454	− ·5273	7·5	+ ·2151	+ ·3914
4·4	− ·2963	− ·4885	7·6	+ ·2529	+ ·3629
4·5	− ·3430	− ·4460	7·7	+ ·2876	+ ·3314
4·6	− ·3853	− ·4003	7·8	+ ·3190	+ ·2970
4·7	− ·4229	− ·3518	7·9	+ ·3469	+ ·2600
4·8	− ·4556	− ·3008	8·0	+ ·3709	+ ·2212
4·9	− ·4831	− ·2482	8·1	+ ·3910	+ ·1806
5·0	− ·5052	− ·1943	8·2	+ ·4070	+ ·1385
5·1	− ·5219	− ·1395	8·3	+ ·4188	+ ·0958
5·2	− ·5331	− ·0816	8·4	+ ·4262	+ ·0526
5·3	− ·5388	− ·0298	8·5	+ ·4293	+ ·0091
5·4	− ·5391	+ ·0241	8·6	+ ·4281	− ·0336
5·5	− ·5340	+ ·0769	8·7	+ ·4227	− ·0751
5·6	− ·5238	+ ·1280	8·8	+ ·4131	− ·1157
5·7	− ·5085	+ ·1769	8·9	+ ·3995	− ·1548
5·8	− ·4885	+ ·2234	9·0	+ ·3821	− ·1921
5·9	− ·4640	+ ·2668	9·1	+ ·3612	− ·2271
6·0	− ·4352	+ ·3070	9·2	+ ·3368	− ·2595
6·1	− ·4027	+ ·3436	9·3	+ ·3094	− ·2889
6·2	− ·3666	+ ·3762	9·4	+ ·2791	− ·3151
6·3	− ·3276	+ ·4046	9·5	+ ·2463	− ·3381
6·4	− ·2858	+ ·4288	9·6	+ ·2117	− ·3574
6·5	− ·2420	+ ·4484	9·7	+ ·1755	− ·3729
6·6	− ·1963	+ ·4633	9·8	+ ·1373	− ·3846
6·7	− ·1495	+ ·4736	9·9	+ ·0982	− ·3924
6·8	− ·1018	+ ·4791	10·0	+ ·0589	− ·3962
6·9	− ·0538	+ ·4799	10·1	+ ·0192	− ·3962
7·0	− ·0060	+ ·4760	10·2	− ·0202	− ·3931

ON CAUCHY'S CONDENSATION TEST FOR THE CONVERGENCY OF SERIES.

By *M. J. M. Hill, M.A., D.Sc., F.R.S.*, Professor of Mathematics at University College, London.

CAUCHY'S "Condensation Test for the Convergency of Series" is as follows:

Let $f(n)$ be a one-valued, continuous function of n which is positive and diminishes as n increases, so that the limit when n is infinite of $f(n)$ is zero, then the series $\Sigma f(n)$ and $\Sigma a^n f(a^n)$ are both convergent or both divergent, where a is any positive integer not less than 2.

It was proved by Kohn in his paper "On the Convergency of Infinite Series," published in 1882 in the 67th part of *Grunert's Archiv*, that the theorem holds if a have any positive integral or fractional value greater than unity, the proof depending on a more general theorem based on geometrical considerations.

The proof here presented is entirely algebraic.

It depends on the following theorems:—

If a be greater than 1, if s be any integer so great that $a^{s+1} - a^s > 1$, and if t_s be the integer next below a^s, then

$$\sum_{n=1+t_s}^{n=\infty} f(n) < a(a - 1 + a^{-s}) \sum_{n=s}^{n=\infty} a^n f(a^n) \quad \ldots\ldots(1).$$

$$\sum_{n=1+t_s}^{n=\infty} f(n) > \frac{1}{a}(1 - a^{-s} - a^{-s-1}) \sum_{n=s+1}^{n=\infty} a^n f(a^n) \quad \ldots(2).$$

1. It follows from the definition of t_s that

$$t_s < a^s < 1 + t_s \quad \ldots\ldots(3),$$

$$t_{s+1} < a^{s+1} < 1 + t_{s+1} \quad \ldots\ldots(4).$$

Since $$a^{s+1} - a^s > 1 \quad \ldots\ldots(5),$$

therefore $$t_{s+1} > t_s \quad \ldots\ldots(6),$$

$$t_{s+1} \nless 1 + t_s \quad \ldots\ldots(7),$$

therefore $$a^{s+1} > 1 + t_s \quad \ldots\ldots(8),$$

$$a^s < t_{s+1} \quad \ldots\ldots(9).$$

2. The following inequalities are required:

$$(1 - a^{-1} - a^{-s-1})\, t_{s+1} < t_{s+1} - t_s \text{ ....(10)},$$

$$t_{s+1} - t_s < (a - 1 + a^{-s})(1 + t_s) \text{ ....(11)}.$$

To prove (10),

$$t_{s+1} > a^{s+1} - 1 \text{ by (4)},$$

$$t_s < a^s \text{ by (3)};$$

therefore $$t_{s+1} - t_s > a^{s+1} - a^s - 1,$$

$$t_{s+1} < a^{s+1} \text{ by (4)};$$

therefore $$\frac{t_{s+1} - t_s}{t_{s+1}} > \frac{a^{s+1} - a^s - 1}{a^{s+1}};$$

therefore $$(1 - a^{-1} - a^{-s-1})\, t_{s+1} < t_{s+1} - t_s.$$

To prove (11),

$$t_{s+1} < a^{s+1} \text{ by (4)},$$

$$1 + t_s > a^s \text{ by (3)};$$

therefore $$t_{s+1} - t_s - 1 < a^{s+1} - a^s,$$

therefore $$t_{s+1} - t_s < a^{s+1} - a^s + 1;$$

but $$1 + t_s > a^s \text{ by (3)};$$

therefore $$\frac{t_{s+1} - t_s}{1 + t_s} < \frac{a^{s+1} - a^s + 1}{a^s};$$

therefore $$t_{s+1} - t_s < (a - 1 + a^{-s})(1 + t_s).$$

3. To prove (1),

$$\begin{aligned}\sum_{n=1+t_s}^{n=\infty} f(n) &= [f(1 + t_s) + \ldots + f(t_{s+1})] \\ &+ [f(1 + t_{s+1}) + \ldots + f(t_{s+2})] \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots \\ &< (t_{s+1} - t_s) f(1 + t_s) + (t_{s+2} - t_{s+1}) f(1 + t_{s+1}) + \ldots \\ &< (a - 1 + a^{-s})(1 + t_s) f(1 + t_s) \\ &+ (a - 1 + a^{-s-1})(1 + t_{s+1}) f(1 + t_{s+1}) \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

Now since $a > 1$,

$$a - 1 + a^{-s-1} < a - 1 + a^{-s},$$
$$a - 1 + a^{-s-2} < a - 1 + a^{-s},$$

and so on; therefore

$$\sum_{n=1+t_s}^{n=\infty} f(n) < (a - 1 + a^{-s}) \left[(1 + t_s) f(1 + t_s) \right.$$
$$\left. + (1 + t_{s+1}) f(1 + t_{s+1}) + \ldots\right].$$

Now $\qquad 1 + t_s > a^s$ by (3),

therefore $\qquad f(1 + t_s) < f(a^s)$,

$\qquad 1 + t_s < a^{s+1}$ by (8),

therefore $\qquad (1 + t_s) f(1 + t_s) < a^{s+1} f(a^s).$

In like manner

$$(1 + t_{s+1}) f(1 + t_{s+1}) < a^{s+2} f(a^{s+1}),$$

and so on; therefore

$$\sum_{n=1+t_s}^{n=\infty} f(n) < a(a - 1 + a^{-s}) \left[a^s f(a^s) + a^{s+1} f(a^{s+1}) + \ldots\right],$$

therefore $\displaystyle \sum_{n=1+t_s}^{n=\infty} f(n) < a(a - 1 + a^{-s}) \sum_{n=s}^{n=\infty} a^n f(a^n),$

which is (1).

4. To prove (2),

$$\sum_{n=1+t_s}^{n=\infty} f(n) = \left[f(1 + t_s) + \ldots + f(t_{s+1})\right]$$
$$+ \left[f(1 + t_{s+1}) + \ldots + f(t_{s+2})\right]$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$> (t_{s+1} - t_s) f(t_{s+1}) + (t_{s+2} - t_{s+1}) f(t_{s+2}) + \ldots$$
$$> (1 - a^{-1} - a^{-s-1})\, t_{s+1} f(t_{s+1})$$
$$+ \ldots + (1 - a^{-1} - a^{-s-2})\, t_{s+2} f(t_{s+2}).$$

by (10).

Now $\qquad 1 - a^{-1} - a^{-s-2} > 1 - a^{-1} - a^{-s-1},$

$\qquad 1 - a^{-1} - a^{-s-3} > 1 - a^{-1} - a^{-s-1},$

and so on; therefore

$$\sum_{n=1+t_s}^{n=\infty} f(n) > (1 - a^{-1} - a^{-s-1}) \left[t_{s+1} f(t_{s+1}) + t_{s+2} f(t_{s+2}) + \ldots\right].$$

Now $$t_{s+1} < a^{s+1} \text{ by } (4),$$

therefore $$f(t_{s+1}) > f(a^{s+1}).$$

Also $$t_{s+1} > a^s \text{ by } (9),$$

therefore $$t_{s+1} f(t_{s+1}) > a^s f(a^{s+1}).$$

In like manner

$$t_{s+2} f(t_{s+2}) > a^{s+1} f(a^{s+2}),$$

and so on; therefore

$$\sum_{n=1+t_s}^{n=\infty} f(n) > \frac{1}{a}(1 - a^{-1} - a^{-s-1})\,[a^{s+1} f(a^{s+1}) + a^{s+2} f(a^{s+2}) + \ldots],$$

therefore $$\sum_{n=1+t_s}^{n=\infty} f(n) > \frac{1}{a}(1 - a^{-1} - a^{-s-1}) \sum_{n=s+1}^{n=\infty} a^n f(a^n).$$

which is (2).

5. The quantity $a - 1 + a^{-s}$ is positive, since $a > 1$, the quantity $1 - a^{-1} - a^{-s-1}$ is positive by (5).

It follows immediately from (1) and (2) that both series are convergent or both divergent.

AN EXHIBITION OF THE COMPLETENESS OF THE SYSTEMS OF FOUR AND FIVE IRREDUCIBLE INVARIANTS OF THE BINARY QUINTIC AND THE BINARY SEXTIC.

By *E. B. Elliott.*

§ 1. An invariant of the binary quantic

$$(a_0, a_1, a_2, \ldots, a_p)(x, y)^p \;\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

is known to be a gradient (rational integral homogeneous isobaric function) in the coefficients, whose degree i and weight w are connected with p by the relation $ip = 2w$, and which is annihilated by the differential operator

$$a_0 \frac{\partial}{\partial a_1} + 2a_1 \frac{\partial}{\partial a_2} + \ldots + pa_{p-1} \frac{\partial}{\partial a_p} \;\ldots\ldots\ldots\ldots(2).$$

An immediate consequence of this annihilation is well known to be that the terms, if any, in an invariant (or any

seminvariant) which are free from $a_0, a_1, \ldots, a_{k-1}$ have the annihilator

$$(k+1)\, a_k \frac{\partial}{\partial a_{k+1}} + \ldots + pa_{p-1} \frac{\partial}{\partial a_p} \ldots\ldots\ldots\ldots(3).$$

To this theorem there proves to be a series of restricted converse theorems with regard to gradients in $a_k, a_{k+1}, \ldots, a_p$, annihilated by (3), which are the parts free from $a_0, a_1, \ldots, a_{k-1}$ in invariants of the p^{ic} (1). Of these theorems the first two are as follows.

I. Case of $k=1$ and $p=2$, 3, or 4. *The gradient solutions of*

$$\left(2a_1 \frac{\partial}{\partial a_2} + \ldots + pa_{p-1} \frac{\partial}{\partial a_p}\right) u = 0,$$

whose degree and weight satisfy the relation $ip = 2w$, for the values 2, 3, 4 *severally of p, which are the parts free from a_0 in invariants of the quadratic, cubic, and quartic respectively, are exactly* (1) *all those of even weight, and* (2) *those of odd weight which contain no term free from a_p.*

An examination of the known forms of invariants of the quadratic, cubic, and quartic at once shows that this is the case.

II. Case of $k=2$ and $p=4$, 5, or 6. *The gradient solutions of*

$$\left(3a_2 \frac{\partial}{\partial a_3} + \ldots + pa_{p-1} \frac{\partial}{\partial a_p}\right) u = 0,$$

whose degree and weight satisfy the relation $ip = 2w$, for the values 4, 5, 6 *severally of p, which are the parts free from a_0, a_1 in invariants of the quartic, quintic, and sextic respectively, are exactly* (1) *all those of even weight and above the first degree, and* (2) *those of odd weight which contain no terms free from both a_{p-1} and a_p.*

This fact will be made evident as we go on.

The next converse theorem of the series will deal with the case of $k=3$ and $p=6$, 7, or 8. It will, I believe, prove to have the analogous simple form with regard to the sextic, septimic, and octavic, and the first three and last three coefficients in them severally, as I. with regard to the quadratic, cubic, and quartic, and the first one and last one coefficient in them severally. I have not, however, applied sufficient tests in the case of the septimic to state this definitely as a fact.

If this series of theorems (and in particular II., with which alone we are concerned in the limits of the present paper) could be proved from the theory of the differential operators, the power of the method which follows would be greatly enhanced, and in particular the exhibition here given of the completeness of the invariant systems for the quintic and sextic would be made independent of appeal to previous knowledge of systems to be tested for completeness, or of the avoidance of such appeal by independent processes at particular stages. But my efforts in this direction have at present failed.

It will be noticed that the gradients which in I. and II., and also in the unproved next theorem of the series, are excluded as not belonging to invariants are exactly those whose rejection is necessitated by the two well-known facts (α) that no invariant can be of the first degree, and (β) that no invariant of odd weight can contain a term symmetrical about the middle of the series of coefficients, since such a term cannot be equal to $(-1)^w$ times itself when w is odd, as it would have to be, did it occur, in virtue of the possible transformation $x=Y$, $y=X$ of modulus -1. Such exclusions do not of necessity suffice for values of p higher than those which go with the different values of k as above; not, for instance, forthe quintic with $k=1$.

§ 2. Let us now confine attention to the case of $k=2$. The question arises whether, when we have a gradient in a_2, a_3, ... which constitutes the part free from a_0, a_1 in an invariant, the invariant of which it is a part is or is not unique.

It is at once answered. If there be two invariants with the same part free from a_0, a_1, their difference is an invariant which vanishes when $a_0=0$ and $a_1=0$, *i.e.* which vanishes for a quantic having two equal roots, *i.e.* which vanishes when the discriminant vanishes, and which consequently has the discriminant for a factor. Now for the quartic (which we shall not further consider), the quintic, and the sextic the respective discriminants are of degrees

$$2(4-1)=6,\ 2(5-1)=8,\ 2(6-1)=10.$$

Thus, for instance, for the sextic an invariant of degree less than 10 with given terms free from a_0, a_1 is unique, two of degree 10 with such given terms differ by a numerical multiple of the discriminant, and two of degree exceeding 10 with such given terms differ by the product of a discriminant and another invariant, so that one of them is reducible in terms of the other and invariants of lower degrees.

Into methods for obtaining the whole expressions of invariants when the parts of them free from a_0, a_1 are known it is not necessary here to enter.

§ 3. *The Quintic.*

Let us now adopt the more convenient alphabetical notation for the coefficients, and write the quintic $(a, b, c, d, e, f)(x, y)^5$.

The terms free from a, b in an invariant are annihilated by

$$3c\frac{\partial}{\partial d} + 4d\frac{\partial}{\partial e} + 5e\frac{\partial}{\partial f},$$

or, in other words, they constitute a seminvariant of the cubic

$$10cx^3 + 10dx^2y + 5exy^2 + fy^3,$$

and are of degree and weight connected by $5i = 2w$.

Let $c^m d^n e^p f^q$ be any one of the terms. The condition $5i = 2w$ gives us

$$5(m + n + p + q) = 2(2m + 3n + 4p + 5q),$$

i.e.
$$m = n + 3p + 5q.$$

Now of this Diophantine equation the simple sets of solutions are

m	n	p	q
1	1		
3		1	
5			1.

All sets of solutions in positive integers are results of adding together multiples of these simple sets. Thus $c^m d^n e^p f^q$ is necessarily of the form $(cd)^r (c^3e)^s (c^5f)^t$, where every one of r, s, t is a positive integer or zero. We require then to examine such solutions of

$$\left(3c\frac{\partial}{\partial d} + 4d\frac{\partial}{\partial e} + 5e\frac{\partial}{\partial f}\right)u = 0$$

as are rational integral functions of

$$cd \equiv \xi, \quad c^3e \equiv \eta, \quad c^5f \equiv \zeta.$$

The differential equation may be written

$$\left(3\frac{\partial}{\partial \xi} + 4\xi\frac{\partial}{\partial \eta} + 5\eta\frac{\partial}{\partial \zeta}\right)u = 0,$$

the two solutions of lowest degrees of which are

$$3\eta - 2\xi^2, \quad 27\zeta - 45\xi\eta + 20\xi^3;$$

and of these no fractional or irrational function can be rational and integral in ξ, η, ζ, as otherwise, putting $\xi = 0$, a fractional or irrational function of 3η and 27ζ would be rational and integral in η and ζ.

Thus the part free from a, b in any invariant of the quintic is a rational integral function of

$$u_4 \equiv 3c^3e - 2c^2d^2,$$

$$u_6 \equiv 27c^5f - 45c^4de + 20c^3d^3.$$

We have to see what rational integral functions of these are the terms free from a, b in invariants, and to employ § 2 to decide the question of the uniqueness of the invariants to which such of them as appertain to invariants belong. The examination for reducibility of the parts free from a, b in all invariants in terms of the parts free from a, b in simple invariants effects for us the examination for reducibility of all invariants in terms of the simple invariants specified and the discriminant, in virtue of § 2.

The conclusions in order are as follows:

(i) There is no invariant of odd degree; for the degrees of u_4 and u_6 are even.

(ii) There is no invariant of degree 2; for u_4, u_6 are of degrees which exceed 2.

(iii) If there is an invariant of degree 4 there can be only one, and its part free from a, b is u_4. Now by reciprocity a quintic must have an invariant of degree 4, since a quantic has one of degree 5, the product IJ. The first irreducible invariant is then precisely given by.

$$\text{the part free from } a, b \text{ in } \left.\begin{matrix} u_4, \\ \tfrac{1}{16}I_4, \end{matrix}\right\} \quad \text{..........................} \quad (4),$$

to use the ordinary notation.

(iv) If there is an invariant of degree 6, it must be unique and have u_6 for its part free from a, b. But u_6 contains the term $20c^3d^3$ which cannot occur in any invariant, by the fact given at the end of § 1 that an invariant of odd weight—the weight of c^3d^3 is 15—cannot contain any term symmetrical about the middle of the series of coefficients. Thus there is no invariant of degree 6.

(v) For a like reason no invariant can have an odd power of u_6 for its part free from a, b, in virtue of the impossibility of an invariant of odd weight containing a term which is a power of cd.

(vi) The only rational integral function of u_4, u_6 whose degree is 8 is u_4^2. One invariant with u_4^2 as its part free from a, b is the square of that specified in (iii) above. There is another in virtue of § 2 above; for 8 is the degree of the discriminant of the quintic, an invariant with no terms free from a, b. There is consequently an irreducible invariant of degree 8; and this may be taken either as the discriminant

$$\Delta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

itself, or as a second invariant with part free from a, b

$$\left.\begin{array}{r} u_4^2. \\ \text{In ordinary notation this is} \qquad \\ \tfrac{1}{2}I_8 \equiv \tfrac{1}{256}I_4^2, \text{ mod. } \Delta. \end{array}\right\} \ldots\ldots\ldots\ldots(5').$$

Any other invariant of degree 8 is a sum of multiples of I_4^2 and I_8, or of I_4^2 and Δ. It is, of course, immaterial whether we regard I_4 and Δ as the two irreducible invariants up to degree 8 or, as is usually done, I_4 and I_8. The former course is somewhat the more convenient from our present point of view.

(vii) Were there an invariant of degree 10 its part free from a, b would have to be u_4u_6. But this product is of odd weight and contains the term $(cd)^5$. As in (v) there can be no invariant containing this term.

(viii) The terms in u_4, u_6 of degree 12 are u_4^3 and u_6^2. The former appertains to an invariant which is the cube of (4), and more generally to invariants of the form $(4)^3 + k\,(4)\,\Delta$, where k is an arbitrary constant. If there is a third invariant of degree 12, linearly independent of $(4)^3$ and $(4)\,\Delta$, there must be one ended by u_6^2, or by $u_6^2 + k'u_4^3$, where k' is arbitrary; and in virtue of § 2 there cannot be a fourth asyzygetic with the third and $(4)^3$ and $(4)\,\Delta$. The easiest way to see that the third exists, and so avoid the necessity of assuming the fact as known from other sources, is to use the fact that

$$u_6^2 + 50u_4^3$$

is a numerical multiple of c^8 times the discriminant of the cubic

$$10cx^3 + 10dx^2y + 5exy^2 + fy^3,$$

for it is c^8 times an invariant of this cubic which vanishes when $c = 0$ and $d = 0$.

Let α, β, γ be the roots of this cubic regarded as one in y/x. Then $u_6^2 + 50u_4^3$ is a numerical multiple of

$$c^{12}\left(\frac{1}{\beta} - \frac{1}{\gamma}\right)^2 \left(\frac{1}{\gamma} - \frac{1}{\alpha}\right)^2 \left(\frac{1}{\alpha} - \frac{1}{\beta}\right)^2,$$

i.e. of $$f^{12}(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2\alpha^8\beta^8\gamma^8.$$

Alter now the meaning of α, β, γ, and let them mean three of the roots of the quintic in y/x,

$$(a, b, c, d, e, f)(x, y)^5,$$

which becomes the above cubic multiplied by y^2 when $a = 0$, $b = 0$; and let δ, ε be the two other roots of the quintic. The function

$$f^{12}\Sigma(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2 \times (\alpha-\delta)^4(\alpha-\varepsilon)^4(\beta-\delta)^4(\beta-\varepsilon)^4(\gamma-\delta)^4(\gamma-\varepsilon)^4$$

is an invariant of the quintic, as the type product in the summation contains every root in 12 factors, and reduces to a single product, namely a numerical multiple of $u_6^2 + 50u_4^3$ as above, when $\delta = 0$, $\varepsilon = 0$, *i.e.* when $a = 0$, $b = 0$, for, since all differences but $\delta - \varepsilon$ occur in the type product written, and since α, β, γ occur symmetrically, every term in the summation which is not the same when $\delta = 0$, $\varepsilon = 0$ as

$$(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2\alpha^8\beta^8\gamma^8$$

contains $\delta - \epsilon$ for a factor, and so vanishes when $\delta = 0$, $\epsilon = 0$.

Accordingly there is a third—an irreducible—invariant of degree 12, whose terms free from a, b may be taken as being either

$$u_6^2 + 50u_4^3,$$

or $$u_6^2 \quad\text{.............................}(6),$$

or $$\lambda u_6^2 + \mu u_4^3.$$

The I_{12} usually taken has for its terms free from a, b

$$-\tfrac{1}{27}(u_6^2 - 4u_4^3)\quad\text{....................}(6').$$

(ix) No invariant whose degree is higher than 12 and divisible by 4 can be irreducible. For if it have no terms free from a, b it is, by § 2, divisible by Δ and is therefore reducible. If, on the other hand, it have a part free from a, b, that part must be of the form

$$(\lambda_0 u_6^{2m} + \lambda_1 u_6^{2m-2} u_4^{3} + \ldots + \lambda_m u_4^{3m})\, u_4^{n},$$

for numerical or zero values of $\lambda_0, \lambda_1, \ldots, \lambda_m$, for an integral or zero value of m, and for one of the values 0, 1, 2 of n. Now such an expression is reducible in terms of (4) and (6). Consequently the invariant consists of a part reducible in terms of I_4 and I_{12}, together with perhaps a part which, by § 2, is the product of Δ and an invariant of lower degree, to which the same reasoning applies.

(x) We proceed now to possible invariants of odd weights, and so of degrees even but not divisible by 4, which alone have still to be considered. If such an invariant have no part free from a, b it is reducible, being the product of a power of Δ and an invariant of odd weight of which Δ is no longer a factor, and which consequently cannot vanish when $a = 0$, $b = 0$. We have then only to pay attention to invariants which have parts free from a, b. These parts, being of oddly even degree, must have u_6 for a factor, as u_4 is evenly even. Now u_6 contains the term $20c^3d^3$. The complementary factor of the part free from a, b in any invariant such as we are considering cannot then contain any term which is a mere power of cd, or the product would contain a term which is an odd power of cd, as it must not by reasoning as in (iv) and (v). Accordingly, if there be any skew invariant with terms free from a, b, those terms must be the product of u_6 and a function of

$$u_4 \equiv 3c^3e - 2c^2d^2, \quad u_6 \equiv 27c^5f - 45c^4de + 20c^3d^3$$

which contains no term that is a power of cd, *i.e.* a function which vanishes, when

$$u_4 = -2c^2d^2, \quad u_6 = 20c^3d^3,$$

i.e. a function with

$$u_6^2 + 50u_4^3$$

for a factor.

Now $$u_6\,(u_6^2 + 50u_4^3)$$

is the part free from a, b in a skew invariant. To see this, and not appeal to entraneous knowledge of the existence of

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an I_{18}, we may proceed as in (viii) above. $u_6^2 + 50u_4^3$ is as there a numerical multiple of

$$f^{12}(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2\alpha^8\beta^8\gamma^8,$$

where α, β, γ are the y/x roots of the cubic there written; and u_6 is, by the seminvariant theory of the cubic, a numerical multiple of

$$c^6\Sigma\left(\frac{1}{\beta}-\frac{1}{\gamma}\right)^2\left(\frac{1}{\gamma}-\frac{1}{\alpha}\right),$$

i.e. a numerical multiple of

$$f^6\Sigma(\beta-\gamma)^2(\alpha-\gamma)\alpha^5\beta^3\gamma^3.$$

Thus $u_6(u_6^2 + 50u_4^3)$ is a numerical multiple of

$$f^{18}\Sigma(\alpha-\beta)^2(\alpha-\gamma)^3(\beta-\gamma)^4\alpha^{13}\beta^{11}\gamma^{11}.$$

Passing now to the notation explained for the quintic, and writing down hence

$$f^{18}\Sigma(\alpha-\beta)^2(\alpha-\gamma)^3(\beta-\gamma)^4$$
$$\times(\alpha-\delta)^6(\alpha-\varepsilon)^7(\beta-\delta)^6(\beta-\varepsilon)^6(\gamma-\delta)^6(\gamma-\varepsilon)^5,$$

we see readily that this does not vanish, but becomes a numerical multiple of the above expression for $u_6(u_6^2 + 50u_4^3)$ in the particular case when $\delta = 0$, $\varepsilon = 0$, *i.e.* when $a = 0$, $b = 0$. It is also an invariant, as the type product contains every root in 18 factors.

Thus $\qquad u_6(u_6^2 + 50u_4^3)$ ....(7)

is the part free from a, b in a skew invariant. In fact in the ordinary notation

$$u_6(u_6^2 + 50u_4^3) + M(a, b) \equiv 27 I_{18} \text{.........}(7').$$

These terms then specify uniquely the first skew—an irreducible —invariant.

(xi) Lastly, there is no other skew irreducible invariant. For, after the opening of (x), we have only to see that no other with terms free from a, b can be irreducible: and the terms free from a, b in any higher skew invariant must be of the form

$$u_6(u_6^2 + 50u_4^3)\,\{\text{function of } u_6^2 \text{ and } u_4\},$$

which are reducible by (4), (6), and (7). The supposed invariant differs then from an invariant specified in reduced form by (4), (6), and (7), if at all, only by the product of a power of Δ and another invariant. In other words, it is reducible.

Thus we have fully exhibited the known fact that in I_4, Δ, I_{12}, I_{18}, or, as is the same thing, in I_4, I_8, I_{12}, and I_{18} is comprised the entire system of irreducible invariants of the binary quintic.

§ 4. *The Sextic.*

The terms free from a, b in an invariant of the sextic

$$(a, b, c, d, e, f, g)(x, y)^6$$

are annihilated by

$$3c\frac{\partial}{\partial d}+4d\frac{\partial}{\partial e}+5e\frac{\partial}{\partial f}+6f\frac{\partial}{\partial g};$$

or, in other words, constitute a seminvariant of the quartic

$$15cx^4+20dx^3y+15ex^2y^2+6fxy^3+gy^4,$$

and are of degree and weight connected by $6i=2w$, *i.e.* $3i=w$.

Let $c^m d^n e^p f^q g^r$ be one of the terms. The degree and weight condition gives

$$m=0.n+p+2q+3r,$$

an equation having for its simple sets of solutions

m	n	p	q	r
	1			
1		1		
2			1	
3				1,

the terms sought are then rational and integral in

$$d\equiv\xi,\ ce\equiv\eta,\ c^2f\equiv\zeta,\ c^3g\equiv\omega,$$

and satisfy $\left(3\frac{\partial}{\partial\xi}+4\xi\frac{\partial}{\partial\eta}+5\eta\frac{\partial}{\partial\zeta}+6\zeta\frac{\partial}{\partial\omega}\right)u=0.$

They are then rational integral functions of

$$u_2 \equiv 3ce - 2d^2,$$

$$u_3 \equiv 27c^2f - 45cde + 20d^3,$$

$$u_4 \equiv 4c^3g - 8c^3df + 5c^2e^2.$$

By § 2 there is no invariant without terms free from a, b of degree less than 10, the only one of degree 10 is the discriminant, and any one of degree exceeding 10 is the product of the discriminant and another invariant.

As to invariants with terms free from a, b, which terms are, as above, rational and integral functions of u_2, u_3, u_4, we have the following succession of observations.

(i) There is no invariant of odd degree less than 10. (Notice that, since $3i = w$, degree and weight are odd or even together, so that invariants of odd degree are skew.)

That invariants of degrees 3, 5 do not exist we are told by reciprocity; for neither a cubic nor a quintic has any invariant of degree 6. Thus u_3 and u_2u_3 do not end invariants.

As to invariants of degree 7, 9, their terms free from a, b would have to be included in the forms

$$u_3(\lambda u_4 + \mu u_2^2),$$

$$u_2u_3(\lambda u_4 + \mu u_2^2) + \nu u_3^3,$$

respectively; and it is at once seen that λ, μ, ν cannot be so chosen so as to make the coefficients of

$$c^3de^3,\ c^2d^3e^2,\ cd^5e,\ d^7,$$

in the first case, all vanish, or those of the terms

$$c^4de^4,\ c^3d^3e^3,\ c^2d^5e^2,\ cd^7e,\ d^9$$

in the second case. Now such terms cannot occur in an invariant, by the last remark in § 1, since they are of odd weight and symmetrical about the middle of the series of coefficients.

(ii) A quadratic has an invariant of degree 6, the cube of its discriminant. Consequently our sextic has an invariant of degree 2. The part free from a, b must be u_2. We have then a first irreducible invariant

$$u_2 + M(a, b) \text{ ....} (8)$$

$$= \tfrac{1}{6}I_2.$$

(iii) If there is an invariant of degree 4 irreducible in terms of (8), there must be one with u_4 for its part free from a, b, since $\lambda u_4 + \mu u_2^2$ is the most general rational integral function of u_2, u_3, u_4, whose degree is 4. Now there is one by reciprocity; for a quartic has two invariants I^3 and J^2 of degree 6. Thus there is a second irreducible invariant

$$u_4 + M(a, b) \text{ ....(9).}$$

The catalecticant, usually taken as I_4, has for its terms free from a, b,

$$-\tfrac{1}{4}(u_4 - u_2^2).$$

(iv) Of degree 6, u_2^3 and u_2u_4 are reducible by (8) and (9). If there be an irreducible invariant of degree 6, there must be an invariant ended by u_3^2, and this or any one ended by terms free from a, b of the form $u_3^2 + \lambda u_2u_4 + \mu u_2^3$ may be taken as the third irreducible invariant.

There is known to be such an invariant I_6. Perhaps the easiest way of proving the fact in connexion with the present theory is to notice that

$$2u_3^2 - 135u_2u_4 + 100u_2^3$$

is a numerical multiple of c^3 times the catalecticant J of the quartic,

$$15cx^4 + 20dx^3y + 15ex^2y^2 + 6fxy^3 + gy^4,$$

i.e. of $\quad c^6\Sigma\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)^2\left(\frac{1}{\alpha}-\frac{1}{\gamma}\right)\left(\frac{1}{\delta}-\frac{1}{\beta}\right)\left(\frac{1}{\delta}-\frac{1}{\gamma}\right)^2,$

i.e. of $\quad g^6\Sigma(\alpha-\beta)^2(\alpha-\gamma)(\delta-\beta)(\delta-\gamma)^2\alpha^3\beta^3\gamma^3\delta^3,$

where α, β, γ, δ are the roots of the cubic regarded as one in y/x, and that the deduced invariant of degree 6 of the sextic

$$g^6\Sigma(\alpha-\beta)^2(\alpha-\gamma)(\delta-\beta)(\delta-\gamma)^2(\alpha-\varepsilon)^2(\alpha-\zeta)$$
$$\times(\beta-\varepsilon)(\beta-\zeta)^2(\gamma-\varepsilon)^2(\gamma-\zeta)(\delta-\varepsilon)(\delta-\zeta)^2$$

does not vanish, but becomes a numerical multiple of the above when $\delta = 0$, $\varepsilon = 0$. This is clear, for the type product here vanishes when any two roots are equal except in the cases of the pairs (ε, ζ), (α, δ), (β, γ), and in the cases of these three pairs respectively, it takes exactly the form above in the sets of four roots $(\alpha, \beta, \gamma, \delta)$, $(\beta, \zeta, \epsilon, \gamma)$, $(\alpha, \varepsilon, \zeta, \delta)$ respectively.

Accordingly there is an irreducible invariant of degree 6. We may take it as the above, or as the one of the form

$$u_3^2 + M(a, b) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

differing from it by multiples of (8) (9) and $(8)^2$. The sextic invariant I_6 usually taken has for its part free from $a, b,$

$$-\tfrac{1}{27}(u_2^3 - 27u_2u_4 + 23u_3^2).$$

(v) Of degree 8 there is no irreducible invariant. For

$$8 = 2 + 2 + 2 + 2 = 2 + 2 + 4 = 2 + 3 + 3 = 4 + 4$$

are all the partitions of 8 into parts 2, 3, 4; and the terms $u_2^4, u_2^2u_4, u_2u_3^2, u_4^2$ are all reducible in terms of (8), (9), and (10).

(vi) Of degree 10 there is no irreducible invariant with terms free from a, b. For $u_2^5, u_2^3u_4, u_2^2u_3^2, u_2u_4^2, u_3^2u_4$ are all reducible in terms of (8), (9), and (10). But in virtue of § 2 there is of this degree an invariant—a single one—with no terms free from a, b, namely the discriminant Δ. This is irreducible in terms of those yet specified, for it is easily verified that numerical constants $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ cannot be found so as to make

$$\lambda_1u_2^5 + \lambda_2u_2^3u_4 + \lambda_3u_2^2u_3^2 + \lambda_4u_2u_4^2 + \lambda_5u_3^2u_4$$

vanish identically. We have then a fourth irreducible invariant which is best taken, for present purposes, as being

$$\text{the discriminant } \Delta \text{ itself} \quad\ldots\ldots\ldots\ldots(11).$$

(vii) There is no irreducible invariant of even degree exceeding 10. For if an invariant of higher degree have no terms free from a, b, it must have Δ for a factor, and if it be not a power of Δ must be the product of a power of Δ and an invariant with such terms; and if it have terms free from a, b those terms must be rational integral functions of u_2, u_4, and u_3^2, so that it is reducible in terms of (8), (9), (10), and Δ in consequence of the above and § 2.

(viii) It remains to consider odd degrees above 10, with consequent odd weight above 30. The terms free from a, b in an invariant of this skew character (—one without such terms is reducible by § 2—) must be divisible by u_3, and have for their complementary factor a function of u_2, u_3^2, and u_4 which has no term in ce and d only. This complementary factor must then be such, that on putting in it

$$u_2 = 3ce - 2d^2,\ u_3 = -45cde + 20d^3,\ u_4 = 5c^2e^2,$$

it must vanish identically. It has then for a factor the eliminant of these regarded as functions of ce and d. This eliminant is

$$(2u_3^2 - 135u_2u_4 + 100u_2^3)^2 - 3645u_4^3.$$

The lowest function which can possibly be a skew invariant has therefore for its terms free from a, b

$$u_3\{(2u_3^2 - 135u_2u_4 + 100u_2^3)^2 - 3645u_4^3\}......(12),$$

and is of degree 15.

To see that there actually is such an invariant, without appealing to our knowledge of the fact from other sources, we may notice that (12) is a numerical multiple of c^6u_3 times the discriminant of the quartic

$$15cx^4 + 20dx^3y + 15ex^2y^2 + 6fxy^3 + gy^4,$$

so that it is a numerical multiple of

$$c^{15}\Sigma\left\{\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)^2\left(\frac{1}{\gamma} - \frac{1}{\alpha}\right)\Pi\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)^2\right\},$$

in a notation as to the quartic already used, *i.e.* is a numerical multiple of

$$g^{15}\Sigma(\alpha-\beta)^4(\alpha-\gamma)^3(\alpha-\delta)^2(\beta-\gamma)^2(\beta-\delta)^3(\gamma-\delta)^2\alpha^6\beta^7\gamma^8\delta^9.$$

We now derive an invariant of the sextic by putting for $\alpha^6\beta^7\gamma^8\delta^9$

$$(\alpha-\epsilon)^3(\alpha-\zeta)^3(\beta-\epsilon)^4(\beta-\zeta)^3(\gamma-\epsilon)^4(\gamma-\zeta)^4(\delta-\epsilon)^4(\delta-\zeta)^5$$

and enlarging the summation so that it be a symmetric one in the six roots, and, as in earlier cases, see that this does not vanish, but becomes, when $\epsilon=0$, $\zeta=0$, *i.e.* when $a=0$, $b=0$, a numerical multiple of the expression which is, as above, a numerical multiple of (12).

Accordingly there is a fifth irreducible invariant—the first skew one—given by the terms free from a, b expressed in (12).

(ix) Lastly, no other skew invariant can be irreducible. For, by the above, its terms free from a, b, or the terms free from a, b in the result of removing from it a power of Δ which divides it, are divisible by (12).

That all invariants of the sextic are reducible in terms of five, of which one is skew, is accordingly exhibited.

ON CERTAIN DEFINITE INTEGRALS, SINGLE AND MULTIPLE.

By Professor *E. J. Nanson.*

1. If $w = ax^2 + 2hx + b$, $w' = a'x^2 + 2h'x + b'$, and $ab > h^2$, then

$$\int_{-\infty}^{+\infty} f\left(\frac{w'}{w}\right) \frac{dx}{w} = \tfrac{1}{2}(ab - h^2)^{-\frac{1}{2}} \int_0^{2\pi} f(\alpha \cos^2\theta + \beta \sin^2\theta)\, d\theta,$$

where α, β are the roots of

$$(ab - h^2)\,\lambda^2 - (ab' + a'b - 2hh')\,\lambda + a'b' - h'^2 = 0.$$

For it is known that real quantities l, m, l', m' can be found, so that

$$w = (lx + m)^2 + (l'x + m')^2,$$

$$w' = \alpha\,(lx + m)^2 + \beta\,(l'x + m')^2.$$

Now let $\quad \tan\theta = (lx + m)/(l'x + m'),$

then $\quad \sec^2\theta\, d\theta = (lm' - l'm)\,(l'x + m')^{-2}\, dx,$

therefore $\quad dx/w = (lm' - l'm)^{-1}\, d\theta,$

and since $\quad ab - h^2 = (lm' - l'm)^2,$

the result stated follows at once.

2. By writing $x = \cot\theta$, we obtain

$$\int_0^{2\pi} f\left(\frac{v'}{v}\right) \frac{d\theta}{v} = (ab - h^2)^{-\frac{1}{2}} \int_0^{2\pi} f(\alpha \cos^2\theta + \beta \sin^2\theta)\, d\theta,$$

where $\quad v = a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta,$

$$v' = a'\cos^2 2\theta + 2h'\cos\theta\sin\theta + b'\sin^2\theta.$$

If in this result we suppose $h = 0$, $h' = 0$, we find

$$\int_0^{2\pi} f\left(\frac{a'\cos^2\theta + b'\sin^2\theta}{a\cos^2\theta + b\sin^2\theta}\right) \frac{dx}{a\cos^2\theta + b\sin^2\theta}$$

$$= \frac{1}{\sqrt{(ab)}} \int_0^{2\pi} f\left(\frac{a'}{a}\cos^2\theta + \frac{b'}{b}\sin^2\theta\right) d\theta.$$

3. If in the last equation we write

$$a = (1+e)^2,\ b = (1-e)^2,\ a' = 1+e,\ b' = 1-e,$$

we find

$$\int_0^\pi f\left(\frac{1+e\cos\phi}{1+2e\cos\phi+e^2}\right)\frac{d\phi}{1+2e\cos\phi+e^2}$$

$$= \frac{1}{1-e^2}\int_0^\pi f\left(\frac{1-e\cos\phi}{1-e^2}\right)d\phi,$$

a result due to John Griffiths, *Messenger*, Vol. XIV., p. 190.

Again, if we write $a' = b' = 1$, $a = 1+e$, $b = 1-e$, where $e < 1$, we have

$$\int_0^\pi f\left(\frac{1}{1+e\cos\phi}\right)\frac{d\phi}{1+e\cos\phi} = \frac{1}{\sqrt{(1-e^2)}}\int_0^\pi f\left(\frac{1-e\cos\phi}{1-e^2}\right)d\phi.$$

This result, in the case in which $f(z) = z^{-n}$, is given by Wolstenholme, *Diff. and Int. Calc.*, p. 39.

4. The results given in sections 1 and 2 may be extended to multiple integrals.

Let w, w' be non-homogeneous quadrics in $n-1$ variables $x_1, \ldots, x_{n-1}$, and let w be definite and positive, then

$$\int_{-\infty}^{+\infty}\ldots\int_{-\infty}^{+\infty} f\left(\frac{w'}{w}\right) w^{-\frac{1}{2}n}dx_1\ldots dx_{n-1} = \tfrac{1}{2}\Delta^{-\frac{1}{2}}\int f(\Sigma a_p l_p^2)\,d\omega,$$

where Δ is the discriminant of w, $d\omega$ is an element of the surface of the unit hypersphere of $n-1$ dimensions

$$y_1^2 + \ldots + y_n^2 = 1,$$

$l_1, \ldots, l_n$ are the direction-cosines of the radius through $d\omega$, $a_1, \ldots, a_n$ are the values of λ for which the discriminant of $\lambda w - w'$ vanishes, and the integration with respect to ω extends over the whole surface of the hypersphere.

For it is known that, by a real linear transformation

$$X_p = \Sigma b_{pq} x_q,$$

where $x_n = 1$ and p, q have all values from 1 to n, we can write

$$w = \Sigma X_p^2, \quad w' = \Sigma a_p X_p^2.$$

Now let $\xi_s = X_s / X_n$, where s has all values from 1 to $n-1$.

Then

$$\frac{\partial(\xi_1, \ldots, \xi_{n-1})}{\partial(x_1, \ldots, x_{n-1})} = X_n^{-n} K(X_n, X_1, \ldots, X_{n-1})$$

$$= X_n^{-n}(b_{pq})$$

$$= MX_n^{-n},$$

where M is the modulus of the linear transformation, so that $M^2\Delta = 1$. Hence, denoting the integral to be found by I, we have

$$I = \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} f\left(\frac{w'}{w}\right) w^{-\frac{1}{2}n} dx_1 \ldots dx_{n-1}$$

$$= \Delta^{-\frac{1}{2}} \int_0^1 \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+} f\left(\frac{w'}{w}\right) w^{-\frac{1}{2}n} X_n^{\,n} d\rho\, d\xi_1 \ldots d\xi_{n-1}.$$

where ρ is a new variable introduced for convenience.

Let $$\xi_s = y_s / y_n, \quad \rho = \Sigma y_p^{\,2},$$

then we readily find

$$\frac{\partial(\xi_1, \ldots, \xi_{n-1}, \rho)}{\partial(y_1, \ldots, y_{n-1}, y_n)} = 2y_n^{-n} \Sigma y_p^{\,2}.$$

Hence $$I = 2\Delta^{-\frac{1}{2}} \int \ldots \int f\left(\frac{\Sigma a_p y_p^{\,2}}{\Sigma y_p^{\,2}}\right) (\Sigma y_p^{\,2})^{-\frac{1}{2}(n-2)} dy_1 \ldots dy_n,$$

the limits being given by

$$0 \overline{\gtrless} y_1^{\,2} + \ldots + y_n^{\,2} \overline{\gtrless} 1, \quad 0 \overline{\gtrless} y_n.$$

We may clearly omit the condition $0 \overline{\gtrless} y_n$, provided we strike out the factor 2. Changing to polar coordinates by writing $y_p = rl_p$ we may replace $dy_1 \ldots dy_n$ by $r^{n-1} dr\, d\omega$. Thus

$$I = \Delta^{-\frac{1}{2}} \int_0^1 \int f(\Sigma a_p l_p^{\,2})\, r\, dr\, d\omega$$

$$= \tfrac{1}{2}\Delta^{-\frac{1}{2}} \int f(\Sigma a_p l_p^{\,2})\, d\omega,$$

where the integration with respect to ω extends over the whole surface of the hypersphere.

5. The extension of the theorem in section 2 is as follows. Let v, v' be homogeneous quadratic functions of the direction cosines $l_1, \ldots, l_n$ of the radius through an element $d\omega$ of

the surface of a unit hypersphere in space of n dimensions, then

$$\int f\left(\frac{v'}{v}\right) v^{-\frac{1}{2}n} d\omega = \Delta^{-\frac{1}{2}} \int f(\Sigma a_p l_p^2)\, d\omega,$$

where Δ is the discriminant of v, a_p is a root of the discriminant of $\lambda v - v'$, v is supposed definite and positive, and the integration extends over the whole surface of the hypersphere.

To prove this theorem let c_p, s_p, t_p denote the cosine, sine, and tangent of θ_p, so that we may write

$$l_1 = c_1,\ l_2 = s_1 c_2,\ l_3 = s_1 s_2 c_3,\ \ldots,\ l_n = s_1 s_2 \ldots s_{n-1},$$

and therefore

$$d\omega = s_1^{n-2} s_2^{n-3} \ldots s_{n-2}\, d\theta_1 \ldots d\theta_{n-1}.$$

Hence if $x_p = l_{p+1}/l_1$, we have

$$x_1 = t_1 c_2,\ x_2 = t_1 s_2 c_3,\ x_3 = t_1 s_2 s_3 c_4,\ \ldots,$$

$$x_{n-1} = t_1 s_2 s_3 \ldots s_{n-1},$$

and therefore

$$\frac{\partial(x_1, \ldots, x_{n-1})}{\partial(\theta_1, \ldots, \theta_{n-1})} = \frac{\partial(x_1, \ldots, x_{n-1})}{\partial(t_1, \theta_2, \ldots, \theta_{n-1})} \frac{\partial(t_1, \theta_2, \ldots, \theta_{n-1})}{\partial(\theta_1, \ldots, \theta_{n-1})}$$

$$= t_1^{n-2} s_2^{n-3} s_3^{n-4} \ldots s_{n-2} \sec^2\theta_1$$

$$= s_1^{n-2} s_2^{n-3} \ldots s_{n-2} \sec^n\theta_1.$$

Thus, if we suppose the coefficients in w, w' to be the same as those in v, v' respectively, so that w, w' may be derived from v, v' by replacing $l_1, \ldots, l_{n-1}, l_n$ by $x_1, \ldots, x_{n-1}, 1$, we have

$$\int f\left(\frac{v'}{v}\right) v^{-\frac{1}{2}n} d\omega = 2\int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} f\left(\frac{w'}{w}\right) w^{-\frac{1}{2}n} dx_1 \ldots dx_{n-1}$$

$$= \Delta^{-\frac{1}{2}} \int f(\Sigma a_p l_p^2)\, d\omega \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The quantities $a_1, \ldots, a_n$ are the roots of the equation

$$\Delta\lambda^n - \Delta_1 \lambda^{n-1} + \ldots + (-1)^n \Delta_n = 0 \ldots\ldots\ldots\ldots(2),$$

where Δ is the discriminant of v or w, Δ_n is the discriminant of v' or w', and $\Delta_1, \Delta_2, \ldots, \Delta_{n-1}$ are the invariants intermediate to Δ, Δ_n.

6. If in the general theorem in the last section we suppose v to be a pure quadric and v' to be a perfect square, so that

$$v = \Sigma b_p l_p^2, \quad v' = (\Sigma e_p l_p)^2,$$

we readily find that one of the roots of (2) is $\Sigma e_p^2/b_p, = k^2$ say, and that the remaining roots are all zero. Hence, writing $f(z) = \phi(\sqrt{z})$, we obtain

$$\int \phi\left(\frac{\Sigma e_p l_p}{\sqrt{(\Sigma b_p l_p^2)}}\right)(\Sigma b_p l_p^2)^{-\frac{1}{2}n}\, d\omega = (b_1 \ldots b_n)^{-\frac{1}{2}} \int \phi(kl)\, d\omega.$$

Now we may clearly write

$$l = \cos\theta, \; d\omega = \sin^{n-2}\theta\, d\theta\, d\omega',$$

where $d\omega'$ is an element of the surface of a unit hypersphere of $n-2$ dimensions. Thus

$$\int \phi(kl)\, d\omega = \int_0^\pi \int \phi(k\cos\theta)\sin^{n-2}\theta\, d\theta\, d\omega'$$

$$= \frac{2\pi^{\frac{1}{2}(n-1)}}{\Gamma(\frac{1}{2}n - \frac{1}{2})} \int_0^\pi \phi(k\cos\theta)\sin^{n-2}\theta\, d\theta.$$

Hence

$$\int \phi\left(\frac{\Sigma e_p l_p}{\sqrt{(\Sigma b_p l_p^2)}}\right)(\Sigma b_p l_p^2)^{-\frac{1}{2}n}\, d\omega$$

$$= \frac{2\pi^{\frac{1}{2}(n-1)}}{\Gamma(\frac{1}{2}n - \frac{1}{2})\sqrt{(b_1 \ldots b_n)}} \int_0^\pi \phi(k\cos\theta)\sin^{n-2}\theta\, d\theta.$$

If in this result we put $n = 3$, we obtain a theorem due to Cauchy, *Journal de l'école polytechnique*, Vol. XIX., p. 529.

7. If in the theorem in the last section we make $b_p = 1$, we get

$$\int \phi(\Sigma e_p l_p)\, dw = \frac{2\pi^{\frac{1}{2}(n-1)}}{\Gamma(\frac{1}{2}n - \frac{1}{2})} \int_0^\pi \phi\{\sqrt{(\Sigma e_p^2)}\cos\theta\}\sin^{n-2}\theta\, d\theta,$$

which is Boole's extension of a theorem of Poisson's, *Camb. Math. Journal*, Vol. III., p 283.

8. From the theorem in section 7, we may at once deduce the theorems in Todhunter's *Integral Calculus*, Arts. 280, 281.

Let $$I=\int\ldots\int f(a_1x_1+\ldots+a_nx_n)\,dx_1\ldots dx_n,$$

the limits being given by

$$0 \gtrless x_1^2+\ldots+x_n^2 \gtrless 1 \quad\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

changing to polars, we write

$$x_p=rl_p,\quad dx_1\ldots dx_n=r^{n-1}dr\,d\omega.$$

Thus if $k^2=a_1^2+\ldots+a_n^2$,

$$I=\int_0^1\int f(\Sigma ra_pl_p)\,r^{n-1}dr\,d\omega$$

$$=\frac{2\pi^{\frac{1}{2}(n-1)}}{\Gamma(\frac{1}{2}n-\frac{1}{2})}\int_0^1\int_0^\pi f(kr\cos\theta)\,r^{n-1}\sin^{n-2}\theta\,dr\,d\theta$$

$$=\frac{2\pi^{\frac{1}{2}(n-1)}}{\Gamma(\frac{1}{2}n-\frac{1}{2})}\int_{-1}^{+1}\int_0^{\sqrt{(1-x^2)}} f(kx)\,y^{n-2}dx\,dy$$

$$=\frac{\pi^{\frac{1}{2}(n-1)}}{\Gamma(\frac{1}{2}n+\frac{1}{2})}\int_{-1}^{+1} f(kx)\,(1-x^2)^{\frac{1}{2}(n-1)}\,dx,$$

which is the theorem of Art. 280.

Again, let

$$I=\int\ldots\int f(a_1x+\ldots+a_nx_n)\,\frac{ax_1\ldots ax_n}{\sqrt{(1-x_1^2-\ldots-x_n^2)}},$$

the limits being given by (3). Then if $d\omega$ be the element of the surface of a unit hypersphere of n dimensions, we have

$$I=\tfrac{1}{2}\int f(a_1l_1+\ldots+a_nl_n)\,d\omega$$

$$=\frac{\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)}\int_0^\pi f(k\cos\theta)\sin^{n-1}\theta\,d\theta$$

$$=\frac{\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)}\int_{-1}^{+1} f(kx)\,(1-x^2)^{\frac{1}{2}n-1}dx,$$

which is the theorem of Art. 281.

This theorem may also be proved by changing to polars as in the proof just given of the theorem of Art. 280.

9. If in the general theorem in section 5 we suppose v to be a perfect square, viz. $(\Sigma e_p l_p)^2$ as before, whilst v is adfected, we readily find that

$$\int \phi\left(\frac{\Sigma e_p l_p}{\sqrt{v}}\right) v^{-\frac{1}{2}n} d\omega = \frac{2\pi^{\frac{1}{2}(n-1)}\Delta^{-\frac{1}{2}}}{\Gamma(\frac{1}{2}n-\frac{1}{2})}\int_0^\pi \phi\left\{\sqrt{\left(-\frac{K_1}{\Delta}\right)}\cos\theta\right\}\sin^{n-2}\theta\, d\theta,$$

where K_1 is the determinant obtained by bordering Δ symmetrically with the coefficients $e_1, \ldots, e_n$.

10. Generally, if we suppose the quadric v' to be degenerate and to have lost m orders, then the equation (2) has m of its roots zero, and the remaining $n-m$ roots are given by

$$\Delta\lambda^{n-m} - \Delta_1\lambda^{n-m-1} + \ldots + (-1)^{n-m}\Delta_{n-m} = 0.$$

If these roots be denoted by $a_1, \ldots, a_{n-m}$, then

$$\int f\left(\frac{v'}{v}\right) v^{-\frac{1}{2}n} d\omega = \Delta^{-\frac{1}{2}} \int f(\Sigma a_p l_p^2)\, d\omega,$$

where p has all values from 1 to $n-m$.

Now writing

$$l_1 = c_1,\ l_2 = s_1 c_2,\ \ldots,\ l_{n-m} = s_1 s_2 \ldots s_{n-m-1} c_{n-m},$$

we have

$$d\omega = s_1^{\,n-2} s_2^{\,n-3} \ldots s_{n-m}^{\,m-1}\, d\theta_1 \ldots d\theta_{n-m}\, d\omega',$$

where $d\omega'$ is an element of the surface of a unit hypersphere of $m-1$ dimensions.

Thus, if $m > 1$,

$$\int f\left(\frac{v'}{v}\right) v^{-\frac{1}{2}n} d\omega = \frac{2\pi^{\frac{1}{2}m}\Delta^{-\frac{1}{2}}}{\Gamma(\frac{1}{2}m)} \int_0^\pi \ldots \int_0^\pi f(\Sigma a_p l_p^2)\, s_1^{\,n-2} \ldots s_{n-m}^{\,m-1}\, d\theta_1 \ldots d\theta_{n-m}.$$

If $m = 1$ there is no reduction in the order of the multiple integral.

11. In the case in which $m = n-2$, the quadric v' breaks up into the product of two factors. If these factors be $\Sigma e_p l_p$, $\Sigma e_p' l_p$, then a_1, a_2 are the roots of the quadratic

$$\Delta\lambda^2 - \Delta_1\lambda + \Delta_2 = 0,$$

where Δ_1 is the determinant obtained by bordering Δ horizontally with the coefficients e and vertically with the coefficients e', whilst Δ_2 is the determinant obtained by bordering Δ symmetrically two deep with the coefficients e, e'.

12. If in section 5 we suppose v, v' both pure quadrics, so that

$$v = \Sigma b_p l_p^2, \quad v' = \Sigma b_p' l_p^2,$$

we have

$$\int f\left(\frac{\Sigma b_p' l_p^2}{\Sigma b_p l_p^2}\right)(\Sigma b_p l_p^2)^{-\frac{1}{2}n} d\omega = \frac{1}{\sqrt{(b_1 \ldots b_n)}} \int f\left(\Sigma \frac{b_p'}{b_p} l_p^2\right) d\omega.$$

This is the generalisation of the theorem stated at the end of section 2.

13. Consider in particular the case in which $f(z) = z^m$ in sections 4 and 5. Let

$$I(m) = \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} w' w^{-m-\frac{1}{2}n} dx_1 \ldots dx_{n-1},$$

$$I'(m) = \int v'^m v^{-m-\frac{1}{2}n} d\omega,$$

$$I''(m) = \int (\Sigma a_p l_p^2)^m \, d\omega,$$

then we have

$$2I(m) = I'(m) = \Delta^{-\frac{1}{2}} I''(m),$$

and, if m be negative, v', and therefore also w', must be definite.

14. The value of each of the integrals $I(m)$, $I'(m)$, $I''(m)$ may be found when m is a positive integer by expanding $(\Sigma a_p l_p^2)^m$ and integrating each term. It may easily be shewn that

$$\int l_p^{2m} d\omega = \frac{2\pi^{\frac{1}{2}(n-1)} \Gamma(m + \frac{1}{2})}{\Gamma(m + \frac{1}{2}n)},$$

and hence by equating coefficients in the obvious identity

$$\int (\Sigma c_p l_p)^{2m} d\omega = (\Sigma c_p^2)^m \int l_p^{2m} d\omega,$$

that

$$\int l_p^{2\alpha} l_q^{2\beta} l_r^{2\gamma} \ldots d\omega = \frac{m!}{(2m)!} \frac{(2\alpha)!\,(2\beta)!\,(2\gamma)!\ldots}{\alpha!\,\beta!\,\gamma!\ldots} \frac{2\pi^{\frac{1}{2}(n-1)} \Gamma(m + \frac{1}{2})}{\Gamma(m + \frac{1}{2}n)},$$

where $\qquad \alpha + \beta + \gamma + \ldots = m \ldots\ldots\ldots\ldots\ldots\ldots (4).$

Hence, if we denote the symmetric function $\Sigma a_p{}^\alpha a_q{}^\beta a_r{}^\gamma \ldots$ of the roots of (2) by $S(\alpha, \beta, \gamma, \ldots)$ and write $L(\alpha)$ for $(2\alpha)!/(\alpha!)^2$, we have

$$I''(m) = \frac{2\pi^{\frac{1}{2}(n-1)}\,\Gamma(m+\frac{1}{2})}{\Gamma(m+\frac{1}{2}n)}\,\Sigma\,\frac{L(\alpha)\,L(\beta)\,L(\gamma)\ldots}{L(m)}\,S(\alpha, \beta, \gamma, \ldots),$$

where Σ denotes summation for all the different sets of values of $\alpha, \beta, \gamma, \ldots$ which satisfy (4).

In point of fact, however, it is easier to find the value of $I(m)$ in terms of $\Delta, \Delta_1, \Delta_2$, &c. by differentiating $I(0)$.

15. We have, from section 13,

$$I(0) = \tfrac{1}{2}\Delta^{-\frac{1}{2}}\int d\omega$$

$$= \frac{\pi^{\frac{1}{2}n}\Delta^{-\frac{1}{2}}}{\Gamma(\frac{1}{2}n)}.$$

Now let $w = (a, b, c, \ldots)(x_1, x_2, \ldots, x_{n-1}, 1)^2,$

$$w' = (a', b', c', \ldots)(x_1, x_2, \ldots, x_{n-1}, 1)^2,$$

$$\delta = a'\frac{d}{da} + b'\frac{d}{db} + c'\frac{d}{dc} + \ldots,$$

so that $$\Delta_r = \frac{1}{r!}\,\delta^r\Delta,$$

$$\delta I(m) = -(m+\tfrac{1}{2}n)\,I(m+1).$$

Hence we find that

$$I(m) = \frac{(-1)^m\pi^{\frac{1}{2}n}}{\Gamma(m+\frac{1}{2}n)}\,\delta^m\Delta^{-\frac{1}{2}}$$

$$= \frac{m!\,\pi^{\frac{1}{2}n}}{\Gamma(m+\frac{1}{2}n)}\,\Sigma\,(-1)^{m-r}\,\frac{(2r)!\,\Delta^{-r-\frac{1}{2}}\Delta_1{}^\alpha\Delta_2{}^\beta\Delta_3{}^\gamma\ldots}{2^{2r}r!\,\alpha!\,\beta!\,\gamma!\ldots},$$

where $r = \alpha + \beta + \gamma + \ldots$ and $\alpha, \beta, \gamma, \ldots$ take all zero and positive integral values subject to

$$\alpha + 2\beta + 3\gamma + \ldots = m,$$

16. The process of the last section may be applied to find the value of a more general integral. Thus, let

$$w_r = (a_r, b_r, c_r, \ldots)(x_1, \ldots, x_{n-1}, 1)^2,$$

$$\delta_r = a_r\frac{d}{da} + b_r\frac{d}{db} + c_r\frac{d}{dc} + \ldots,$$

then we have

$$\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} w_1 w_2 \dots w_m w^{-m-\frac{1}{2}n} dx_1 \dots dx_{n-1} = \frac{(-1)^m \pi^{\frac{1}{2}n}}{\Gamma(m+\frac{1}{2}n)} \delta_1 \dots \delta_m \Delta^{-\frac{1}{2}}.$$

Again, we know that

$$\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} e^{-u} dx_1 \dots dx_n = \pi^{\frac{1}{2}n} \Delta^{-\frac{1}{2}}.$$

where u is a definite positive homogeneous quadric in $x_1, \dots, x_n$ which we may without loss of generality suppose to reduce to w when $x_n = 1$, so that Δ has the same meaning as before. Hence

$$\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} u_1 \dots u_m e^{-u} dx_1 \dots dx_n = (-1)^m \pi^{\frac{1}{2}n} \delta_1 \dots \delta_m \Delta^{-\frac{1}{2}},$$

where u_r is the value of w_r when made homogeneous by the introduction of the additional variable x_n.

The values of the two integrals in this section may therefore be expressed in terms of the invariants of the system $u, u_1, \dots, u_m$.

17. Resuming the consideration of the integral $I'(m)$, we have from section 14,

$$\int v^{-\frac{1}{2}n} d\omega = \Delta^{-\frac{1}{2}} \int d\omega$$

$$= \frac{2\pi^{\frac{1}{2}n}\Delta^{-\frac{1}{2}}}{\Gamma(\frac{1}{2}n)}.$$

Hence if $\Delta(\lambda)$ be the discriminant of $v + \lambda\Sigma l_p^2$ and λ be positive, or if negative then numerically less than the minimum value of v, so that $v + \lambda$ is essentially positive, we have

$$\int (v+\lambda)^{-\frac{1}{2}n} d\omega = \frac{2\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n)\sqrt{\{\Delta(\lambda)\}}} \quad \dots\dots\dots\dots(5),$$

and differentiating m times with respect to λ,

$$\int (v+\lambda)^{-\frac{1}{2}n-m} d\omega = \frac{2(-1)^m \pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n+m)} \left(\frac{d}{d\lambda}\right)^m \frac{1}{\sqrt{\{\Delta(\lambda)\}}}.$$

18. Now

$$\int v'^m v^{-m-\frac{1}{2}n} d\omega = \Delta^{-\frac{1}{2}} \int (\Sigma a_p l_p^2)^m d\omega$$

for all values of m. Replacing then m by $-\frac{1}{2}n - m$, where

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the new m is a positive integer and assuming v' to be definite, we have

$$\int v'^{-\frac{1}{2}n-m} v^m d\omega = \Delta^{-\frac{1}{2}} \int (\Sigma a_p l_p^2)^{-\frac{1}{2}n-m} d\omega,$$

and by the last section the value of the second member is the value when $\lambda = 0$ of

$$\frac{2(-1)^m \pi^{\frac{1}{2}n} \Delta^{-\frac{1}{2}}}{\Gamma(\frac{1}{2}n+m)} \left(\frac{d}{dx}\right)^m \frac{1}{\sqrt{\{(a_1+\lambda)\ldots(a_n+\lambda)\}}}.$$

But we have identically

$$\Delta(\lambda+a_1)(\lambda+a_2)\ldots(\lambda+a_n) = \Sigma \Delta_p \lambda^{n-p},$$

where $\Delta_0 = \Delta$. Thus, if m be a positive integer,

$$\int v'^{-\frac{1}{2}n-m} v^m d\omega = \frac{2(-1)^m \pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n+m)} \underset{\lambda=0}{Lt} \left(\frac{d}{d\lambda}\right)^m \frac{1}{\sqrt{(\Sigma \Delta_p \lambda^{n-p})}},$$

and by interchanging v, v', we see that

$$\int v'^m v^{-\frac{1}{2}n-m} d\omega = \frac{2(-1)^m \pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n+m)} \underset{\lambda=0}{Lt} \left(\frac{d}{d\lambda}\right)^m \frac{1}{\sqrt{(\Sigma \Delta_p \lambda^p)}}.$$

It is easy to see that the value thus found for $I'(m)$ is identical with that given by sections 13, 15.

19. Again, integrating (5) with respect to λ between the limits 0 and ∞, we have

$$\int v^{-\frac{1}{2}n+1} d\omega = \frac{2\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n-1)} \int_0^\infty \frac{d\lambda}{\sqrt{\{\Delta(\lambda)\}}},$$

and so, generally, if m be a positive integer $< \frac{1}{2}n$,

$$\int v^{-\frac{1}{2}n+m} d\omega = \frac{2\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n-m)} \int_0^\infty \ldots \int_0^\infty \frac{d\lambda_1 \ldots d\lambda_m}{\sqrt{\Delta(\lambda_1+\ldots+\lambda_m)}}$$

$$= \frac{2\pi^{\frac{1}{2}n}}{\Gamma(m)\,\Gamma(\frac{1}{2}n-m)} \int_0^\infty \frac{\lambda^{m-1} d\lambda}{\sqrt{\{\Delta(\lambda)\}}} \quad \ldots\ldots(6).$$

20. Now proceeding as in section 18 and replacing m by $-\frac{1}{2}n+m$, where the new m is a positive integer $< \frac{1}{2}n$, we have

$$\int v'^{-n\frac{1}{2}+m} v^{-m} d\omega = \Delta^{-\frac{1}{2}} \int (\Sigma a_p l_p^2)^{m-\frac{1}{2}n} d\omega$$

$$= \phi(m) \int_0^\infty \frac{\lambda^{m-1} d\lambda}{\sqrt{(\Sigma \Delta_p \lambda^{n-p})}}$$

$$= \phi(m) \int_0^\infty \frac{\lambda^{\frac{1}{2}n-m-1} d\lambda}{\sqrt{(\Sigma \Delta_p \lambda^p)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(7),$$

where $\phi(m)$ has been written for $2\pi^{\frac{1}{2}n} / \Gamma(m)\,\Gamma(\frac{1}{2}n-m)$.

Also by interchanging v, v', we see that

$$\int v'^{-m} v^{-\frac{1}{2}n+m} d\omega = \phi(m) \int_0^\infty \frac{\lambda^{m-1} d\lambda}{\sqrt{(\Sigma\Delta_p\lambda^p)}} \quad \text{.........(8).}$$

When n is even, formulæ (7), (8) are not distinct.

21. Let us now suppose n to be odd and equal to $2r+1$ and like v' to be Σl_p^2. Formula (8) reduces to (6), but from (7) we have

$$\int v^{-m} d\omega = \phi(m) \int_0^\infty \frac{\lambda^{r-m-\frac{1}{2}} d\lambda}{\sqrt{\{\Delta(\lambda)\}}} \quad \text{...............(9),}$$

where m may have any positive integral value from 1 to r inclusive.

Proceeding as in section 17, we deduce from (9) that

$$\int (v+\mu)^{-m} d\omega = \phi(m) \int_0^\infty \frac{\lambda^{r-m-\frac{1}{2}}\lambda}{\sqrt{\{\Delta(\lambda+\mu)\}}} \quad \text{........(10)}$$

$$= \phi(m) \int_\mu \frac{(\lambda-\mu)^{r-m-\frac{1}{2}}}{\sqrt{\{\Delta(\lambda)\}}} d\lambda \quad \text{.........(11).}$$

Let $r-m=k$, then, differentiating (11) $1, 2, \ldots, k-1$ times, we easily see that we merely introduce the formula (11) with a new m. Further differentiation of (11) is not permissible by the ordinary rule.

22. In (10), let $m=r$; thus

$$\int (v+\mu)^{-r} d\omega = \phi(m) \int_0^\infty \frac{d\lambda}{\sqrt{\{\lambda\Delta(\lambda+\mu)\}}}.$$

Differentiating s times with respect to μ, we get

$$\int (v+\mu)^{-r-s} d\omega = \frac{2(-1)^s \pi^r}{\Gamma(r+s)} \int_0^\infty \left(\frac{d}{d\mu}\right)^s \frac{1}{\sqrt{\{\Delta(\lambda+\mu)}} \frac{d\lambda}{\sqrt{\lambda}}.$$

Making $\mu=0$, we find

$$\int v^{-r-s} d\omega = \frac{2(-1)^s \pi^r}{\Gamma(r+s)} \int_0^\infty \frac{d\lambda}{\sqrt{\lambda}} \left(\frac{d}{d\lambda}\right)^s \frac{1}{\sqrt{\{\Delta(\lambda)\}}} \quad \text{...(12).}$$

Thus when n is odd the value of $\int v^m d\omega$ is known for any integral value of m positive or negative.

33. By means of the theorems of section 5 we may now find the values of $\int v^{\frac{1}{2}m} d\omega$, where m has any odd integral value, positive or negative, n being odd as before.

For $$\int v^{-\frac{1}{2}n-m} d\omega = \Delta^{-\frac{1}{2}} \int (\Sigma a_p l_p^2)^m d\omega,$$

and the value of the second member is known for any integral value of m.

24. Next let n be even and equal to $2r$. When m is a positive integer or zero the value of $\int v^m d\omega$ is known by section 15. If m be a negative integer between 0 and $-r$, the value of $\int v^m d\omega$ is known from (6). If $m = r$, we have

$$\int v^m d\omega = \frac{2\pi^{\frac{1}{2}n} \Delta^{-\frac{1}{2}}}{\Gamma(\frac{1}{2}n)}.$$

If m be a negative integer between $-r$ and $-\infty$, then, by section 13,

$$\int v^m d\omega = \Delta^{-\frac{1}{2}} \int (\Sigma a_p l_p^2)^{-m-r} d\omega,$$

and the second number is known by section 15 because $-m-r$ is a positive integer.

Thus, when n is even, $\int v^m d\omega$ is known when m has any integral value, positive or negative.

25. Some other integrals may be reduced to those considered in this paper.

With the notation of section 16, let

$$I'''(m) = \int \ldots \int u'^m dx_1 \ldots dx_n,$$

the limits being given by $0 \overline{<} u \overline{<} 1$.

Reducing u, u' to the forms Σy_p^2, $\Sigma a_p y_p^2$, where y_p is a linear homogeneous function of $x_1, \ldots, x_n$, and a_p is a root of (2), we obtain

$$I'''(m) = \Delta^{-\frac{1}{2}} \int \ldots \int (\Sigma a_p y_p^2)^m dy_1 \ldots dy_n,$$

where the limits are given by $0 \overline{<} \Sigma y_p^2 \overline{<} 1$.

Replacing $dy_1...dy_n$ by the polar element of volume $r^{n-1}dr\,d\omega$, we get

$$I'''(m) = \Delta^{-\frac{1}{2}} \int_0^1 \int r^{2m+n-1} (\Sigma a_p l_p^2)^m \, dr\, d\omega$$

$$= \frac{1}{2m+n} \Delta^{-\frac{1}{2}} \int (\Sigma a_p l_p^2)^m \, d\omega$$

$$= \frac{1}{m+\frac{1}{2}n} I(m).$$

Thus, if m be zero or a positive integer,

$$I'''(m) = \frac{(-1)^m \pi^{\frac{1}{2}n}}{\Gamma(m+1+\frac{1}{2}n)} \delta^m \Delta^{-\frac{1}{2}}.$$

The cases in which $m=1$ are in effect given by Scott, *Messenger*, Vol. v., pp. 23, 24.

8. Again, let]

$$P(m) = \int p^{-2m+1} dS,$$

where $$dS^2 = (dx_1...dx_n)^2 \left\{\frac{1}{dx_1^2} + ... + \frac{1}{dx_n^2}\right\},$$

$$\frac{4}{p^2} = \left(\frac{du}{dx_1}\right)^2 + ... + \left(\frac{du}{dx_n}\right)^2,$$

and the integration extends over the whole surface of the hyper-ellipsoid $u=0$.

By an orthogonal transformation, let u be reduced to the form $\Sigma a_p y_p^2$. Thus

$$dS^2 = (dy_1...dy_n)^2 \left\{\frac{1}{dy_1^2} + ... + \frac{1}{dy_n^2}\right\},$$

$$\frac{4}{p^2} = \left(\frac{du}{dy_1}\right)^2 + ... + \left(\frac{du}{dy_n}\right)^2$$

$$= 4\Sigma a_p^2 y_p^2,$$

and, if $d\omega$ be an element of the surface of a unit hypersphere of $n-1$ dimensions, we readily find that

$$p\,dS = r^n \, d\omega.$$

Thus

$$P(m) = \int p^{-2m} r^n d\omega$$

$$= \int (\Sigma \alpha_p^2 y_p^2)^m r^n d\omega$$

$$= \int r^{n+2m} (\Sigma \alpha_p^2 l_p^2)^m d\omega$$

$$= (n+2m) \int_0^r \int r^{n+2m-1} (\Sigma \alpha_p^2 l_p^2)^m dr d\omega$$

$$= (n+2m) \int \ldots \int (\Sigma \alpha_p^2 y_p^2)^m dy_1 \ldots dy_n.$$

the limits being given by $0 \overline{<} \Sigma \alpha_p y_p^2 \overline{<} 1$.

Hence, by the last section, if m be a positive integer, we have

$$P(m) = \frac{2(-1)^m \pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n+m)} \delta^m \Delta^{-\frac{1}{2}},$$

where $$\Delta = \alpha_1 \alpha_2 \ldots \alpha_n,$$

$$\delta = \Sigma \beta_p^2 \frac{d}{d\alpha_p},$$

and after the differentiations have been performed the β's are to be replaced by α's. We readily find that

$$P(m) = \frac{2\pi^{\frac{1}{2}n} B_n^{-\frac{1}{2}}}{\Gamma(\frac{1}{2}n+m)} \Sigma \frac{(2r)!\, B_1^{\alpha} B_2^{\beta} B_3^{\gamma} \ldots}{2^{2r} r!\, \alpha!\, \beta!\, \gamma! \ldots}$$

where B_p is the Boolian orthogonal invariant of u of order p, $r = \alpha + \beta + \gamma + \ldots$ and $\alpha, \beta, \gamma, \ldots$ take all zero and positive integral values subject to

$$\alpha + 2\beta + 3\gamma + \ldots = m.$$

The cases in which $m = 0$ and $m = 1$ are in effect given by Scott, *l.c.*

Melbourne.

NOTE ON THE PRINCIPLE OF DUALITY.

By *J. Brill, M.A.*

1. SUPPOSE that we have given a diagram consisting of three fixed points A, B, C, whose coordinates are respectively (x_1, y_1), (x_2, y_2), (x_3, y_3), and a fixed straight line whose equation is

$$Ux + Vy - 1 = 0 \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Suppose also that the three points A, B, C have the respective multiples m_1, m_2, m_3 attached to them. Let AB be joined and produced to meet the given straight line in the point O. Take a point P on AB such that

$$\frac{PB.AO}{AP.BO} = \frac{m_1}{m_2},$$

and attach to the point P the multiple $m_1 + m_2$. Join PC, and produce it to meet (1) in O'. Take a point Q on PC such that

$$\frac{QC.PO'}{PQ.CO'} = \frac{m_1 + m_2}{m_3}.$$

Let (x, y) and $(\bar{x}, \bar{y})$ be the respective coordinates of P and Q.

If we write (X, Y) for the coordinates of O, we readily obtain

$$X = \frac{x_2 - x_1 - V(x_2y_1 - x_1y_2)}{U(x_2 - x_1) + V(y_2 - y_1)},$$

$$Y = \frac{y_2 - y_1 + U(x_2y_1 - x_1y_2)}{U(x_2 - x_1) + V(y_2 - y_1)}.$$

Now, since the ratios of the segments of a straight line are equal to the ratios of their projections on the axis of x, we have

$$\frac{m_1}{m_2} = \frac{(x_2 - x)\left\{\dfrac{x_2 - x_1 - V(x_2y_1 - x_1y_2)}{U(x_2 - x_1) + V(y_2 - y_1)} - x_1\right\}}{(x - x_1)\left\{\dfrac{x_2 - x_1 - V(x_2y_1 - x_1y_2)}{U(x_2 - x_1) + V(y_2 - y_1)} - x_2\right\}}$$

$$= \frac{(x_2 - x)(1 - Ux_1 - Vy_1)}{(x - x_1)(1 - Ux_2 - Uy_2)},$$

since, after reduction, the common factor $(x_2 - x_1)$ may be removed from both the numerator and the denominator.

Solving the above equation for x, we obtain

$$x\{m_1(1 - Ux_2 - Vy_2) + m_2(1 - Ux_1 - Vy_1)\}$$
$$= m_1x_1(1 - Ux_2 - Vy_2) + m_2x_2(1 - Ux_1 - Vy_1).$$

Similarly we should obtain, for the formula giving y, the equation

$$y\{m_1(1 - Ux_2 - Vy_2) + m_2(1 - Ux_1 - Vy_1)\}$$
$$= m_1y_1(1 - Ux_2 - Vy_2) + m_2y_2(1 - Ux_1 - Vy_1).$$

Thus we should also have

$$\bar{x}\{(m_1 + m_2)(1 - Ux_3 - Vy_3) + m_3(1 - Ux - Vy)\}$$
$$= (m_1 + m_2)x(1 - Ux_3 - Vy_3) + m_3x_3(1 - Ux - Vy).$$

Substituting for x and y their values, reducing, and dividing by $m_1 + m_2$, we have

$$\begin{aligned}\bar{x}\{m_1(1 - Ux_2 - Vy_2)&(1 - Ux_3 - Vy_3)\\ &+ m_2(1 - Ux_3 - Vy_3)(1 - Ux_1 - Vy_1)\\ &+ m_3(1 - Ux_1 - Vy_1)(1 - Ux_2 - Vy_2)\}\\ = m_1x_1(1 - Ux_2 - Vy_2)&(1 - Ux_3 - Vy_3)\\ &+ m_2x_2(1 - Ux_3 - Vy_3)(1 - Ux_1 - Vy_1)\\ &+ m_3x_3(1 - Ux_1 - Vy_1)(1 - Ux_2 - Vy_2).\end{aligned}$$

It is evident that we should also obtain a similar formula for $\bar{y}$.

It is to be noted that the formulæ for $\bar{x}$ and $\bar{y}$ are symmetrical in the m's, x's, and y's. Thus there are three methods of fixing the point Q, all leading to the same result.

2. Now suppose that we have given a second diagram consisting of three fixed straight lines, their equations being

$$u_1x + v_1y - 1 = 0 \quad\text{.....................(2)},$$
$$u_2x + v_2y - 1 = 0 \quad\text{.....................(3)},$$
$$u_3x + v_3y - 1 = 0 \quad\text{.....................(4)},$$

and a fixed point whose coordinates are (X, Y). Suppose also that the multiples m_1, m_2, m_3 are respectively attached to the lines (2), (3), (4).

Suppose that the line

$$y - Y = \mu (x - X) \qquad\qquad (5),$$

drawn through the point O, meets (2) and (3) in A and B respectively. Further, suppose a point P taken on this line such that

$$\frac{OA.PB}{AP.OB} = \frac{m_1}{m_2}.$$

For the coordinates of the point A, we readily obtain

$$x = \frac{1 - v_1 (Y - \mu X)}{u_1 + \mu v_1},$$

$$y = \frac{\mu + u_1 (Y - \mu X)}{u_1 + \mu v_1}.$$

The coordinates of B are obtained by substituting u_2 and v_2 for u_1 and v_1 in these formulæ.

Thus, taking (x, y) for the coordinates of P, we have

$$\frac{m_1}{m_2} = \frac{\left\{\frac{1 - v_1 (Y - \mu X)}{u_1 + \mu v_1} - X\right\} \left\{\frac{1 - v_2 (Y - \mu X)}{u_2 + \mu v_2} - x\right\}}{\left\{x - \frac{1 - v_1 (Y - \mu X)}{u_1 + \mu v_1}\right\} \left\{\frac{1 - v_2 (Y - \mu X)}{u_2 + \mu v_2} - X\right\}}.$$

On reducing this with the aid of equation (5), we obtain

$$\frac{(1 - u_1 X - v_1 Y)(1 - u_2 x - v_2 y)}{(1 - u_1 x - v_1 y)(1 - u_2 X - v_2 Y)} = -\frac{m_1}{m_2}.$$

On further reduction this becomes

$$\begin{aligned} x \{m_1 u_1 (1 - u_2 X - v_2 Y) + m_2 u_2 (1 - u_1 X - v_1 Y)\} \\ + y \{m_1 v_1 (1 - u_2 X - v_2 Y) + m_2 v_2 (1 - u_1 X - v_1 Y)\} \\ - m_1 (1 - u_2 X - v_2 Y) - m_2 (1 - u_1 X - v_1 Y) = 0. \end{aligned}$$

The locus of P is therefore a straight line, and, expressing its equation in the form

$$ux + vy - 1 = 0,$$

we have

$$u\{m_1(1 - u_2X - v_2Y) + m_2(1 - u_1X - v_1Y)\}$$
$$= m_1u_1(1 - u_2X - v_2Y) + m_2u_2(1 - u_1X - v_1Y),$$
$$v\{m_1(1 - u_2X - v_2Y) + m_2(1 - u_1X - v_1Y)\}$$
$$= m_1v_1(1 - u_2X - v_2Y) + m_2v_2(1 - u_1X - v_1Y).$$

Thus it is clear that if we attach to this line the multiple $m_1 + m_2$, and from it and (4), to which we have attached the multiple m_3, derive a fresh line in a similar manner; then, expressing the equation of the new line in the form

$$\bar{u}x + \bar{v}y - 1 = 0,$$

we shall obtain

$$\bar{u}\{m_1(1 - u_2X - v_2Y)(1 - u_3X - v_3Y)$$
$$+ m_2(1 - u_3X - v_3Y)(1 - u_1X - v_1Y)$$
$$+ m_3(1 - u_1X - v_1Y)(1 - u_2X - v_2Y)\}$$
$$= m_1u_1(1 - u_2X - v_2Y(1 - u_3X - v_3Y)$$
$$+ m_2u_2(1 - u_3X - v_3Y)(1 - u_1X - v_1Y)$$
$$+ m_3u_3(1 - u_1X - v_1Y)(1 - u_2X - v_2Y),$$

and a similar formula for $\bar{v}$.

It is to be noted that these formulæ are symmetrical in the m's, u's, and v's. Thus there are three different methods of obtaining the final line, all leading to the same result.

3. It will be noticed that the analogy between the results obtained in the two preceding articles is exact. The position of the point Q in the first case will depend upon the ratios of the three quantities m_1, m_2, m_3; and, by the suitable adjustment of these ratios, may be made to coincide with any point in the plane. The same is true of the final line in the second case. We may therefore establish a simple correspondence between the two diagrams, a line in the second diagram

corresponding to a point in the first, the corresponding lines and points being those for which the above-mentioned ratios are identical.

Now suppose that we have given three points in the first diagram, the multiples corresponding to these points being respectively (m_1, m_2, m_3), (m_1', m_2', m_3'), (m_1'', m_2'', m_3''). We proceed to discover the condition that these three points may be collinear. For brevity, we will write

$$1 - Ux_1 - Vy_1 = a,\ 1 - Ux_2 - Vy_2 = b,\ 1 - Ux_3 - Vy_3 = c.$$

Then the required condition will be the vanishing of the determinant

$$\begin{vmatrix} m_1bc + m_2ca + m_3ab, & m_1x_1bc + m_2x_2ca + m_3x_3ab, & m_1y_1bc + m_2y_2ca + m_3y_3ab \\ m_1'bc + m_2'ca + m_3'ab, & m_1'x_1bc + m_2'x_2ca + m_3'x_3ab, & m_1'y_1bc + m_2'y_2ca + m_3'y_3ab \\ m_1''bc + m_2''ca + m_3''ab, & m_1''x_1bc + m_2''x_2ca + m_3''x_3ab, & m_1''y_1bc + m_2''y_2ca + m_3''y_3ab \end{vmatrix}.$$

It is easily seen that this condition reduces to

$$a^2b^2c^2 \begin{vmatrix} 1, & x_1, & y_1 \\ 1, & x_2, & y_2 \\ 1, & x_3, & y_3 \end{vmatrix} . \begin{vmatrix} m_1, & m_2, & m_3 \\ m_1', & m_2', & m_3' \\ m_1'', & m_2'', & m_3'' \end{vmatrix} = 0.$$

We suppose that a, b, c do not vanish, *i.e.* that each of the three points A, B, C lies outside the line (1). We also suppose that A, B, C are not collinear, and therefore form a triangle of finite area. Thus it follows that our condition reduces to

$$\begin{vmatrix} m_1, & m_2, & m_3 \\ m_1', & m_2', & m_3' \\ m_1'', & m_2'', & m_3'' \end{vmatrix} = 0.$$

Considering now the corresponding lines in the second diagram, we will write

$$1 - u_1X - v_1Y = \alpha,\ 1 - u_2X - v_2Y = \beta,\ 1 - u_3X - v_3Y = \gamma;$$

and then we shall obtain for the condition of the concurrency of the three lines, the equation

$$\alpha^2\beta^2\gamma^2 \begin{vmatrix} 1, & u_1, & v_1 \\ 1, & u_2, & v_2 \\ 1, & u_3, & v_3 \end{vmatrix} \cdot \begin{vmatrix} m_1\,, & m_2\,, & m_3 \\ m_1', & m_2', & m_3' \\ m_1'', & m_2'', & m_3'' \end{vmatrix} = 0.$$

If we now suppose that each of the three lines (2), (3), (4) passes clear of the point O, and also that the said three lines are not concurrent, then the above condition reduces to

$$\begin{vmatrix} m_1\,, & m_2\,, & m_3 \\ m_1', & m_2', & m_3' \\ m_1'', & m_2'', & m_3'' \end{vmatrix} = 0.$$

If, therefore, we suppose the above-mentioned restrictions to hold with regard to the fundamental elements of the two diagrams, then to three collinear points in the first diagram there correspond three concurrent lines in the second diagram. Thus to all points on a given line in the first diagram will correspond all lines through a certain point in the second diagram. The said point may be supposed to correspond to the given line. We have, therefore, a correspondence such that to a point and a line through it in the first diagram there correspond a line and a point on it in the second diagram. The correspondence is therefore a reciprocal one.

4. Now suppose that we have two fixed points (f, g) and (h, k) in the first diagram, and let the multiples attached to them be (l_1, l_2, l_3) and (n_1, n_2, n_3). Then the multiples attached to any point in the same straight line with them will be of the form

$$\lambda l_1 + \nu n_1, \quad \lambda l_2 + \nu n_2, \quad \lambda l_3 + \nu n_3.$$

If (x, y) be the coordinates of this point, then we have the three equations

$$bc\,(\lambda l_1 + \nu n_1)\,(x - x_1) + ca\,(\lambda l_2 + \nu n_2)\,(x - x_2)$$
$$+ ab\,(\lambda l_3 + \nu n_3)\,(x - x_3) = 0,$$

$$bcl_1\ (f - x_1) + cal_2\ (f - x_2) + abl_3\ (f - x_3) = 0,$$

$$bcn_1\,(h - x_1) + can_2\,(h - x_2) + abn_3\,(h - x_3) = 0\,;$$

and therefore

$$\begin{vmatrix} (\lambda l_1+\nu n_1)(x-x_1), & (\lambda l_2+\nu n_2)(x-x_2), & (\lambda l_3+\nu n_3)(x-x_3) \\ l_1(f-x_1) \quad , & l_2\,(f-x_2) \quad , & l_3\,(f-x_3) \\ n_1(h-x_1) \quad , & n_2\,(h-x_2) \quad , & n_3\,(h-x_3) \end{vmatrix}=0.$$

Writing μ for ν/λ, this takes the form

$$Ax\mu+Bx+C\mu+D=0.$$

Thus if x, x', x'', x''' be the abscissae of four points on the line joining (f, g) and (h, k), and μ, μ', μ'', μ''' be the values of μ corresponding to them, we have

$$\frac{(x'''-x)(x''-x')}{(x''-x)(x'''-x')}=\frac{(\mu'''-\mu)(\mu''-\mu')}{(\mu''-\mu)(\mu'''-\mu')}.$$

Also, considering the four corresponding lines in the second diagram, we shall obtain

$$\frac{(u'''-u)(u''-u')}{(u''-u)(u'''-u')}=\frac{(\mu'''-\mu)(\mu''-\mu')}{(\mu''-\mu)(\mu'''-\mu')}.$$

Now the values of μ for a point and its corresponding line are identical. Also in order to find the anharmonic ratio of a pencil, we have only to consider the segments that it cuts out of the axis of x. Thus it is clear that the anharmonic ratio of the pencil corresponding to four collinear points is the same as that of the range consisting of the four points themselves.

Now suppose that we have four concurrent lines LP, LQ, LR, LS in the first diagram, and let P, Q, R, S be the points in which they are met by a straight line. Let P', Q', R', S' be the points in the second diagram which correspond to LP, LQ, LR, LS. Then P', Q', R', S' are collinear. Also if L' be the point corresponding to PS, then $L'P'$, $L'Q'$, $L'R'$, $L'S'$ are the lines corresponding to the points P, Q, R, S. Now the anharmonic ratio of the pencil $\{L.PQRS\}$ is the same as that of the range $(PQRS)$. But, by what has been proved above, the anharmonic ratio of the range $(PQRS)$ is equal to that of the pencil $\{L'.P'Q'R'S'\}$, which is the same as that of the range $(P'Q'R'S')$. Thus the anharmonic ratio of the range $(P'Q'R'S')$ is the same as that of the pencil $\{L.PQRS\}$ to which it corresponds.

5. Thus, considered from a sufficiently general point of view, the duality of homoloidal space appears to be complete.

ON LAGRANGE'S PARENTHESES IN THE PLANETARY THEORY.

By *E. T. Whittaker, B.A.*

LET the elements of a planet's orbit be a, the mean distance; e, the eccentricity; i, the inclination of the plane of the orbit; ϵ, the mean longitude at the epoch; ϖ, the longitude of the perihelion; Ω, the longitude of the ascending node.

Let x, y, z; $\dot{x}, \dot{y}, \dot{z}$ be the coordinates and components of velocity of a planet moving in the orbit; and let p, q denote any two of the above elements.

Then, if we write

$$\frac{\partial x}{\partial p}\frac{\partial \dot{x}}{\partial q} - \frac{\partial x}{\partial q}\frac{\partial x'}{\partial p} = \frac{\partial (x, \dot{x})}{\partial (p, q)},$$

we have $$[p, q] = \frac{\partial (x, \dot{x})}{\partial (p, q)} + \frac{\partial (y, \dot{y})}{\partial (p, q)} + \frac{\partial (z, \dot{z})}{\partial (p, q)},$$

where $[p, q]$ is one of Lagrange's parentheses. The fundamental equations of the Planatary Theory are

$$\frac{\partial R}{\partial p} = \sum_q [p, q]\frac{dq}{dt}\ \begin{pmatrix} p = a, e, i, \epsilon, \varpi, \Omega \\ q = a, e, i, \epsilon, \varpi, \Omega \end{pmatrix}.$$

The theorem we shall prove is that

$$[p, q] = \partial\frac{\left(\dfrac{\epsilon - \varpi}{n}, \dfrac{-\mu}{2a}\right)}{\partial (p, q)} + \frac{\partial (\varpi - \Omega, h)}{\partial (p, q)} + \frac{\partial (\Omega, h\cos i)}{\partial (p, q)},$$

where $n = \sqrt{\left(\dfrac{\mu}{a^3}\right)}$, $h = \sqrt{\{a(1 - e^2)\}}$, and μ is the sun's mass.

To prove this result, let

$$x = x'\cos\Omega - y'\sin\Omega,$$
$$y = y'\cos\Omega + x'\sin\Omega,$$
$$z = z'.$$

Then, replacing x, y, z by these expressions, we find

$$[p, q] = \frac{\partial (x', \dot{x}')}{\partial (p, q)} + \frac{\partial (y', \dot{y}')}{\partial (p, q)} + \frac{\partial (z', \dot{z}')}{\partial (p, q)} + \frac{\partial (\Omega, x'\dot{y} - y'x)}{\partial (p, q)}.$$

But the first three terms on the right-hand side are what the parenthesis wonld be if Ω were zero. Denote this by $[p, q]'$.

Also $$x'\dot{y}' - y'\ ' = h\cos i.$$

Therefore $[p, q] = [p, q]' + \partial \dfrac{(\Omega, h \cos i)}{\partial (p, q)}$.

Now put
$$x' = x'',$$
$$y' = y'' \cos i - z'' \sin i,$$
$$z' = z'' \cos i + y'' \sin i.$$

Replacing x', y', z' by these expressions, we find that

$$[p, q]' = \frac{\partial (x'', \dot{x}'')}{\partial (p, q)} + \frac{\partial (y'', \dot{y}'')}{\partial (p, q)} + \frac{\partial (z'', \dot{z}'')}{\partial (p, q)} + \frac{\partial (i, y''\dot{z}'' - z''\dot{y}'')}{\partial (p, q)}.$$

But
$$y''\dot{z}'' - z''\dot{y}'' = 0.$$

So, if $[p, q]''$ denote what the parenthesis would be if Ω and i were zero, we have

$$[p, q]' = [p, q]''$$

so
$$[p, q] = [p, q]'' + \frac{\partial (\Omega, h \cos i)}{\partial (p, q)}.$$

Now, let
$$x'' = x''' \cos (\varpi - \Omega) - y''' \sin (\varpi - \Omega),$$
$$y'' = y''' \cos (\varpi - \Omega) + x''' \sin (\varpi - \Omega),$$
$$z'' = z'''.$$

Replacing x'', y'', z'', by these values, we find

$$[p, q]'' = [p, q]''' + \frac{\partial (\varpi - \Omega,\ \dot{y}'''x''' - y'''\dot{x}''')}{\partial (p, q)},$$

where now $[p, q]'''$ denotes what the parenthesis would be if Ω, i, and $\varpi - \Omega$, were zero, *i.e.* if the axes were the major axis, the latus rectum, and the normal to the orbit.

But
$$\dot{y}'''x''' - y'''\dot{x}''' = h,$$

so
$$[p, q] = [p, q]''' + \frac{\partial (\varpi - \Omega,\ h)}{\partial (p, q)} + \frac{\partial (\Omega,\ h \cos i)}{\partial (p, q)}.$$

Now z''' is permanently zero. So putting

$$x''' = X,\quad y''' = Y,$$

X and Y are functions of a, e, and $nt + \epsilon - \varpi$, only.

Write $\dfrac{\epsilon - \varpi}{n} = k$. Then as n is a function of a only, X and Y are functions of a, e, and $t + k$, only; and

$$\dot{X} = \frac{\partial X}{\partial k},\quad \ddot{X} = \frac{\partial \dot{X}}{\partial k}.$$

Therefore, by differentiation, we find that

$$\frac{\partial(X,\dot{X})}{\partial(p,q)}+\frac{\partial(Y,\dot{Y})}{\partial(p,q)}=\frac{\partial(a,e)}{\partial(p,q)}\left[\frac{\partial(X,\dot{X})}{\partial(a,e)}+\frac{\partial(Y,\dot{Y})}{\partial(a,e)}\right]$$
$$+\frac{\partial(e,k)}{\partial(p,q)}\left[\frac{\partial(X,\dot{X})}{\partial(e,k)}+\frac{\partial(Y,\dot{Y})}{\partial(e,k)}\right]$$
$$+\frac{\partial(k,a)}{\partial(p,q)}\left[\frac{\partial(X,\dot{X})}{\partial(k,a)}+\frac{\partial(Y,\dot{Y})}{\partial(k,a)}\right].$$

Now, as t does not occur explicitly in Lagrange's parentheses, they will always be equal to their values at the point where $t+k=0$. But in the vicinity of this point, we have the expansions

$$X=a(1-e)-\frac{\mu}{2a^2(1-e^2)}(t+k)^2+\ldots,$$

$$Y=\sqrt{\left(\frac{\mu}{a}\,\frac{1+e}{1-e}\right)}(t+k)+\ldots.$$

From these we find that, at the point $t+k=0$,

$$\frac{\partial X}{\partial a}=1-e,\quad \frac{\partial Y}{\partial a}=0,\quad \frac{\partial \dot{X}}{\partial a}=0,\quad \frac{\partial \dot{Y}}{\partial a}=-\frac{1}{2a}\sqrt{\left(\frac{\mu}{a}\,\frac{1+e}{1-e}\right)},$$

$$\frac{\partial X}{\partial e}=-a,\quad \frac{\partial Y}{\partial e}=0,\quad \frac{\partial \dot{X}}{\partial e}=0,\quad \frac{\partial \dot{Y}}{\partial e}=-\sqrt{\left(\frac{\mu}{a(1-e^2)}\right)}\frac{1}{1-e},$$

$$\frac{\partial X}{\partial k}=0,\quad \frac{\partial Y}{\partial k}=\sqrt{\left(\frac{\mu}{a}\,\frac{1+e}{1-e}\right)},\quad \frac{\partial \dot{X}}{\partial k}=-\frac{\mu}{a^2(1-e)^2},\quad \frac{\partial \dot{Y}}{\partial k}=0.$$

Substituting these values, we find that

$$\frac{\partial(X,\dot{X})}{\partial(a,e)}+\frac{\partial(Y,\dot{Y})}{\partial(a,e)}=0,$$

$$\frac{\partial(X,\dot{X})}{\partial(e,k)}+\frac{\partial(Y,\dot{Y})}{\partial(e,k)}=0,$$

$$\frac{\partial(X,\dot{X})}{\partial(k,a)}+\frac{\partial(Y,\dot{Y})}{\partial(k,a)}=\frac{\mu}{2a^2},$$

Therefore

$$[p,q]'''=\frac{\partial(k,a)}{\partial(p,q)}\,\frac{\mu}{2a^2}=\frac{\partial\left(k,-\dfrac{\mu}{2a}\right)}{\partial(p,q)}.$$

Therefore

$$[p, q] = \frac{\partial\left(\dfrac{\varepsilon - \varpi}{n}, -\dfrac{\mu}{2a}\right)}{\partial(p, q)} + \frac{\partial(\varpi - \Omega, h)}{\partial(p, q)} + \frac{\partial(\Omega, h\cos i)}{\partial(p, q)},$$

so the theorem is proved.

This result gives the simplest method of actually evaluating the fifteen Lagrange's parentheses.

For instance, to find $[a, \varepsilon]$, we have

$$[a, \varepsilon] = \frac{\partial\left(\dfrac{\varepsilon - \varpi}{n}, -\dfrac{\mu}{2a}\right)}{\partial(a, e)} = -\frac{\mu}{2a^2}\,\frac{\partial\left(\dfrac{\varepsilon - \varpi}{n}\right)}{\partial e} = -\frac{\mu}{2a^2 n}$$

$$= -\tfrac{1}{2}\sqrt{\left(\frac{\mu}{a}\right)}.$$

In this way we can find all the parentheses, and deduce the fundamental equations of the Planetary Theory.

This result holds also when p and q are any two elements selected from a set of six equivalent to $a, e, i, \varepsilon, \varpi, \Omega$; and so furnishes us with a set of canonical elements for the orbit; namely, the quantities

$$\frac{\varepsilon - \varpi}{n}, \; -\frac{\mu}{2a}, \; \varpi - \Omega, \; h, \; \Omega, \; h\cos i.$$

For, if these be written $\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3$ respectively, we have

$$[\alpha_1, \beta_1] = 1, \quad [\alpha_2, \beta_2] = 1, \quad [\alpha_3, \beta_3] = 1,$$

and all the other parentheses are zero. This proves that $\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3$ are a canonical set of elements, and that the fundamental equations can be written in the form

$$\frac{d\alpha_1}{dt} = \frac{\partial R}{\partial \beta_1}, \quad \frac{d\alpha_2}{dt} = \frac{\partial R}{\partial \beta_2}, \quad \frac{d\alpha_3}{dt} = \frac{\partial R}{\partial \beta_3},$$

$$\frac{d\beta_1}{dt} = -\frac{\partial R}{\partial \alpha_1}, \quad \frac{d\beta_2}{dt} = -\frac{\partial R}{\partial \alpha_2}, \quad \frac{d\beta_3}{dt} = -\frac{\partial R}{\partial \alpha_3}.$$

In this way we can dispense with the methods of integration which are usually given for finding the set of canonical constants $\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3$.

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SOME HYPERBOLOIDS CONNECTED WITH A TETRAHEDRON.

By G. R. R. Routh.

1. THE four perpendiculars from the vertices of a tetrahedron on the opposite faces lie on a hyperboloid of one sheet.

This can be proved by the methods of solid geometry, but more easily by statical considerations; for by resolving along an edge we see that four forces which act at the corners and are proportional and perpendicular to the opposite faces are in equilibrium. They are therefore generators of a hyperboloid.

2. Let $ABCD$ be the tetrahedron, and AK, BL, CM, DN the perpendiculars on the opposite faces, then the projections of AK, BL, CM on face ABC are the lines through A, B, C perpendiculars to the opposite sides, and therefore meet in the orthocentre P. Hence a line through P perpendicular to the face ABC meets AK, BL, CM, and therefore is a generator of the opposite system.

3. A sphere centre D meets the faces DAB, DCA, DBC in a spherical triangle, and the planes DAK, DBL, DCM in three arcs through the vertices and perpendicular to the opposite sides of this spherical triangle. These three arcs meet in the spherical orthocentre S; hence the line DS meets the three lines AK, BL, CM, and therefore is a generator of the hyperboloid.

4. The tetrahedral equation to the quadric referred to $ABCD$ is found to be

$$\frac{zu}{CD}(\cos\widehat{xz}\cos\widehat{yu}-\cos\widehat{xu}\cos\widehat{yz})\cos\widehat{xy}+\ldots.$$

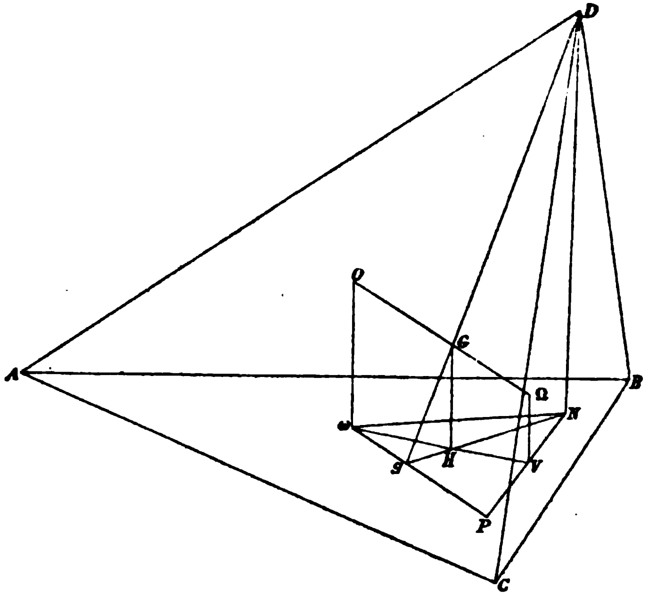
The equations to DN are

$$\frac{x}{A\cos\widehat{xu}}=\frac{y}{B\cos\widehat{yu}}=\frac{z}{C\cos\widehat{zu}}.$$

The other generator through D, being the intersection of the planes AKD, BLD, is given by

$$\frac{x}{A}\cos\widehat{yz}=\frac{y}{B}\cos\widehat{zx}=\frac{z}{C}\cos\widehat{xy}.$$

5. Let O be the centre of the circumscribing sphere and ω the circumcentre of the base ABC. Let G be the centre of gravity of the tetrahedron and g of the base, then $DG = 3Gg$. Draw GH perpendicular to the base, then H lies in gN and $gN = 4gH$. Let V be the middle point of PN.



Now, the ratio $\dfrac{gH \,.\, NV \,.\, P\omega}{HN \,.\, VP \,.\, \omega g} = \frac{1}{3} \,.\, 1 \,.\, 3 = 1$, and therefore the three points ω, H, V lie on a straight line.

Hence, if a straight line be drawn through V perpendicular to the base, it meets OG, say in Ω.

Again, if we draw gX parallel to NP to meet ωV in X, we have

$$\frac{\omega X}{XV} = \frac{\omega g}{gP} = \frac{1}{2} \frac{XH}{XV} = \frac{gH}{HN} = \frac{1}{3};$$

therefore $$\frac{\omega H}{XV} = \frac{3}{4} = \frac{HN}{gN} = \frac{HV}{XV},$$

therefore $\omega H = HV$. Therefore also $OG = G\Omega$.

6. The lines through P and N perpendicular to the face ABC are parallel generators. Therefore the centre of the hyperboloid lies in $V\Omega$. But it must also lie in the lines

obtained in a similar manner from the other faces, all of which pass through Ω. Hence, Ω is the centre of the hyperboloid.

The above theorems appear in Salmon's *Solid Geometry*, but no proof is given of (5) and (6).

7. Let Q be any point of the hyperboloid, join OQ and take Q' in OQ, so that $OQ' = \frac{1}{3} OQ$, then Q' traces out a new hyperboloid. This hyperboloid clearly contains the perpendiculars to the faces of the tetrahedron through their centres of gravity for $\omega g = \frac{1}{3}\omega P$.

If N' be taken in ωN, so that $\omega N' = \frac{1}{3} ON$, then the line through N' perpendicular to face ABC is also a generator. Hence, by a proof similar to that for the first hyperboloid, its centre Ω' lies in OG, and is such that $O\Omega' = \frac{1}{3} O\Omega$.

Since $OQ' = \frac{1}{3} OQ$, and $O\Omega' = \frac{1}{3} O\Omega$, it is clear that $\Omega' Q'$ is parallel to ΩQ and equals $\frac{1}{3}\Omega Q$. Therefore the two hyperboloids are similar and similarly situated, and their linear dimensions are in the ratio 3 to 1.

Again, if QG be produced to q, so that $Gq = \frac{1}{3} QG$, we get the second hyperboloid again. Hence, O and G are external and internal centres of similitude of the two hyperboloids.

8. In OQ take a point q', so that $Oq' = \frac{1}{m} OQ$, then we can show in a similar way that q' lies on a similar and similarly situated hyperboloid, that its centre lies in OG, say at Ω'', and is such that $O\Omega'' = \frac{1}{m} O\Omega$.

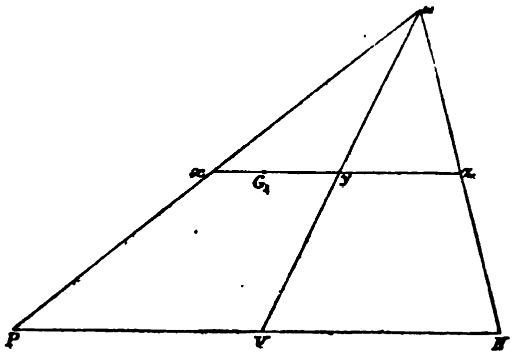
This hyperboloid and the first hyperboloid have O for external, and a point S in OG for internal, centre of similitude, where $\Omega'' S = \frac{1}{m} S\Omega$.

It contains the straight lines perpendicular to base which divide ωN, ωP in ratio 1 to m.

9. Let us now place four particles mass p at the circumcentres of the faces, four of mass q at the orthocentres, and four of mass r at the feet of the perpendiculars from the opposite corners. Let $G_1 G_2 G_3 G_4$ be the centres of gravity of the three particles in each face, then the perpendiculars to the faces through them lie on a hyperboloid for four forces acting along them proportional to the faces are in equilibrium.

Draw a parallel through G_4 to meet ωP in x, ωV in y, and ωN in z, then we easily find $\frac{\omega x}{\omega P} = \frac{q + r}{p + q + r}$; there-

fore the perpendicular to ABC through x lies on a hyperboloid similar to the hyperboloids previously obtained, and its centre lies in $O\Omega$ at Y, so that $\frac{OY}{O\Omega} = \frac{q+r}{p+q+r}$. We can also easily obtain that $\frac{yG_4}{xy} = \frac{q-r}{q+r} =$ a constant. Hence it is



clear that the hyperboloid containing the lines through $G_1 G_2 G_3 G_4$ perpendicular to their respective faces as generators is similar and similarly situated to the hyperboloid containing the perpendicular to ABC through x and the corresponding lines. Hence it is also similar and similarly situated to the original hyperboloid.

Its centre is Y, and by principles of symmetry we see that if q and r be interchanged, the hyperboloid is unchanged.

If, however, $q = r$, then and then only the four lines meet in a point.

10. Among others of these hyperboloids we may notice the hyperboloid containing the perpendiculars through the nine-point centres of the faces. The centre of this hyperboloid is the centre of gravity of the tetrahedron.

11. The coordinates of Ω and the equation to the hyperboloid can be found referred to the three medians as oblique axes.

For this we require to know the length and the equation of the perpendicular from the origin G on the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

Let $x'y'z'$ be the coordinates of the foot of the perpendicular, and let $\lambda\mu\nu$ be the cosines of the angles between the axes.

Projecting on the three axes in turn, we have

$$x' + y'\nu + z'\mu = p\frac{p}{a},$$

$$x'\nu + y' + z'\lambda = p\frac{p}{b},$$

$$x'\mu + y'\lambda + z' = p\frac{p}{c},$$

and
$$\frac{x'}{a} + \frac{y'}{b} + \frac{z'}{c} = 1,$$

whence, eliminating $x'y'z'$, we obtain

$$\frac{1}{p^2}\begin{vmatrix} 1, \nu, \mu \\ \nu, 1, \lambda \\ \mu, \lambda, 1 \end{vmatrix} + \begin{vmatrix} 1, & \nu, & \mu, & \frac{1}{a} \\ \nu, & 1, & \lambda, & \frac{1}{b} \\ \mu, & \lambda, & 1, & \frac{1}{c} \\ \frac{1}{a}, & \frac{1}{b}, & \frac{1}{c}, & 0 \end{vmatrix} = 0;$$

and its equation is

$$aX = bY = cZ,$$

where
$$X = x + y\nu + z\mu,$$
$$Y = x\nu + y + z\lambda,$$
$$Z = x\mu + y\lambda + z.$$

12. Take G for origin and the lines joining G to the middle points of DA, DB, DC for axes, and let their lengths be a, b, c.

Now the equation to the plane DAB is

$$\frac{x}{a} + \frac{y}{b} - \frac{z}{c} = 1,$$

therefore the equation to the line through G perpendicular to it is $aX = bY = -cZ$. Therefore, since C is the point $-a$, $-b$, c, the equation to the perpendicular from C is

$$a\{X - (-a - b\nu + c\mu)\} = b\{Y - (-a\nu - b + c\lambda)\}$$
$$= c\{Z - (-a\mu - b\lambda + c)\}.$$

This can be simplified to

$$-aX-bc\lambda-a^2=-bY-ca\mu-b^2=cZ+ab\lambda-c^2;$$

or, denoting $aX+bc\lambda$ by $a\xi$,
$bY+ca\mu$ by $b\eta$,
$cZ+ab\lambda$ by $c\zeta$,

the equation becomes

$$-a\xi-a^2=-b\eta-b^2=c\zeta-c^2.$$

Similarly, the other perpendiculars can be found to be

$$\begin{aligned} -a\xi-a^2 &= b\eta-b^2=-c\zeta-c^2, \\ a\xi-a^2 &= -b\eta-b^2=-c\zeta-c^2, \\ a\xi-a^2 &= b\eta-b^2=c\zeta-c^2. \end{aligned}$$

Consider the equation

$$La^2\xi^2+Mb^2\eta^2+Nc^2\zeta^2=1.$$

It contains the above lines if

$$L(\lambda+a^2)^2+M(\lambda+b^2)^2+N(\lambda+c^2)^2=1,$$

for all values of λ,

i.e.

$$\begin{aligned} L+M+N&=0, \\ La^2+Mb^2+Nc^2&=0, \\ La^4+Mb^4+Nc^4&=1. \end{aligned}$$

Solving these equations, we obtain

$$L=\frac{1}{(a^2-b^2)(a^2-c^2)},\ M=\frac{1}{(b^2-c^2)(b^2-a^2)},\ N=\frac{1}{(c^2-a^2)(c^2-b^2)}.$$

Therefore the equation to the hyperboloid is

$$\frac{a^2\xi^2}{(a^2-b^2)(a^2-c^2)}+\frac{b^2\eta^2}{(b^2-c^2)(b^2-a^2)}+\frac{c^2\zeta^2}{(c^2-a^2)(c^2-b^2)}=1.$$

Corollary. The coordinates of Ω are given by

$$X=-\frac{bc}{a}\lambda,\ \ Y=-\frac{ca}{b}\mu,\ \ Z=-\frac{ab}{c}\nu,$$

and the coordinates of O are identical, but with opposite signs.

The lengths of the perpendiculars from Ω and O on the four faces give the quadriplanar coordinates of Ω and O.

DEMONSTRATION RELATIVE A L'INVERSEUR DE HART.

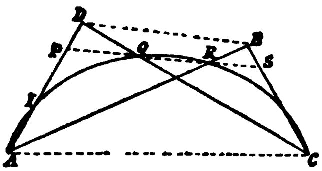
Par le Prof. *Mannheim.*

L'INVERSEUR bien connu dû à Mr. Hart se compose de quatre tiges articulées AB, BC, CD, AD. Les tiges AB, CD sont d'égales longueurs, il en est de même de AD et BC.

Quelle que soit la déformation de la figure formée par les quatre tiges, les droites AC, BD restent parallèles.

Pour une position des tiges menons la droite P, Q, R, S parallèlement à AC.

Les points P, Q, R, S étant marqués sur les tiges qui les contiennent, on a toujours $PQ \times PR =$ const. quelle que soit la position des tiges. Il s'agit d'établir cette propriété.



Par les points A, Q, R, C on peut faire passer une circonférence de cercle; décrivons cette courbe, elle coupe AD au point I.

On a $DI \times DA = DQ \times DC$, donc le point I est le même sur AD, quelle que soit la position des tiges.

On a maintenant $PQ \times PR = PI \times PA$.

Le produit qui est au second membre de cette égalité est constant, d'après ce qui vient d'être remarqué, donc $PQ \times PR$ est un produit constant. Ce qu'il fallait démontrer.

On peut voir de la même manière que $QP \times QS =$ const

Il m'a paru utile de faire connaître cette courte démonstration, que j'emploie depuis plusieurs années dans mon *Cours à l'Ecole polytechnique*, et qui n'a pas encore été publiée.

ON THE DEFINITE INTEGRALS CONNECTED WITH THE BERNOULLIAN FUNCTION.

By *J. W. L. Glaisher.*

Introduction, § 1.

§ 1. The object of the present paper is to consider the definite integrals which occur in connection with certain expansions relating to the Bernoullian Function. The formulæ required are taken from a somewhat lengthy paper on the Bernoullian Function now in course of publication in *Quarterly Journal.** They are readily intelligible as they stand, and recourse need only be had to the paper in the *Quarterly Journal* for the demonstrations or for collateral matters which are not essential in relation to the discussion of the integrals.

It may be premised that the quantities $A_n'(x)$ and $A_n(x)$ are rational and integral functions of x of the orders $n-1$ and n respectively, and that

$$A_n'(x) = A_n(x) - 2^n A_n(\tfrac{1}{2}x).$$

Expressions for $A'_{2n+1}(x)$ and $A'_{2n}(x)$ as definite integrals, §§ 2–4.

§ 2. In § 204 of the paper in the *Quarterly Journal*, the following formulæ relating to the functions $A_n'(x)$ are given:

$$\text{(i)}\ \sin\pi x + \frac{\sin 3\pi x}{3^{2n+1}} + \frac{\sin 5\pi x}{5^{2n+1}} + \&\text{c.} = (-1)^n \tfrac{1}{2}\frac{\pi^{2n+1}}{(2n)!}A'_{2n+1}(x),$$

$$\text{(ii)}\ \cos\pi x + \frac{\cos 3\pi x}{3^{2n}} + \frac{\cos 5\pi x}{5^{2n}} + \&\text{c.} = (-1)^n \tfrac{1}{2}\frac{\pi^{2n}}{(2n-1)!}A'_{2n}(x),$$

$$\text{(iii)}\ \tfrac{1}{2}a\frac{\cos(x-\frac{1}{2})a}{\cos\frac{1}{2}a} = aA_1'(x) - \frac{a^3}{2!}A_3'(x) + \frac{a^5}{4!}A_5'(x) - \&\text{c.},$$

$$\text{(iv)}\ \tfrac{1}{2}a\frac{\sin(x-\frac{1}{2})a}{\cos\frac{1}{2}a} = a^2A_2'(x) - \frac{a^4}{3!}A_4'(x) + \frac{a^6}{5!}A_6'(x) - \&\text{c.},$$

$$\text{(v)}\ \frac{ae^{ax}}{e^a+1} = aA_1'(x) + a^2A_2'(x) + \frac{a^3}{2!}A_3'(x) + \frac{a^4}{3!}A_4'(x) + \&\text{c.}$$

* "On the Bernoullian Function," *Quarterly Journal,* Vol. XXIX., p. 1.

In equation (i) x must lie between the limits 0 and 1. In equation (ii) x must not exceed these limits. Equations (iii), (iv), (v) hold good for all values of x.

§ 3. In order to express the series in (i) as a definite integral, we notice that

$$e^{-u}\sin\pi x + e^{-3u}\sin 3\pi x + e^{-5u}\sin 5\pi x + \&c. = \frac{(e^{u}+e^{-u})\sin\pi x}{e^{2u}+e^{-2u}-2\cos 2\pi x},$$

and that, n being a positive integer,

$$\int_0^\infty u^n e^{-ku}\,du = \frac{n!}{k^{n+1}}.$$

Multiplying therefore the series by u^{2n} and integrating, we have

$$(2n)!\left\{\sin\pi x + \frac{\sin 3\pi x}{3^{2n+1}} + \frac{\sin 5\pi x}{5^{2n+1}} + \&c.\right\}$$
$$= \sin\pi x\int_0^\infty \frac{u^{2n}(e^{u}+e^{-u})\,du}{e^{2u}+e^{-2u}-2\cos 2\pi x},$$

and therefore from (i)

$$(-1)^n\,\tfrac{1}{2}\pi^{2n+1}A'_{2n+1}(x) = \sin\pi x\int_0^\infty \frac{u^{2n}\cosh u\,du}{\cosh 2u - \cos 2\pi x}.$$

In a similar manner we derive from equation (ii) the formula

$$(-1)^n\,\tfrac{1}{2}\pi^{2n}A'_{2n}(x) = \cos\pi x\int_0^\infty \frac{u^{2n-1}\sinh u\,du}{\cosh 2u - \cos 2\pi x}.$$

§ 4. Transforming the integrals by putting $u=\pi t$, these formulæ become

$$\text{(i)}\quad A'_{2n+1}(x) = (-1)^n\,2\sin\pi x\int_0^\infty \frac{t^{2n}\cosh\pi t\,dt}{\cosh 2\pi t - \cos 2\pi x},$$

$$\text{(ii)}\quad A'_{2n}(x) = (-1)^n\,2\cos\pi x\int_0^\infty \frac{t^{2n-1}\sinh\pi t\,dt}{\cosh 2\pi t - \cos 2\pi x};$$

or, putting the integrals on the left-hand side of the equations,

$$\text{(iii)}\quad \int_0^\infty \frac{t^{2n}\cosh\pi t\,dt}{\cosh 2\pi t - \cos 2\pi x} = (-1)^n\frac{A'_{2n+1}(x)}{2\sin\pi x},$$

$$\text{(iv)} \qquad \int_0^\infty \frac{t^{2n-1} \sinh \pi t \, dt}{\cosh 2\pi t - \cos 2\pi x} = (-1)^n \frac{A'_{2n}(x)}{2 \cos \pi x}.$$

In (i) and (iii) x must lie between the limits 0 and 1; in (ii) and (iv) it must not exceed these limits.

Evaluation of general integrals, §§ 5–7.

§ 5. Substituting for $A'_{2n+1}(x)$ and $A'_{2n}(x)$, from (i) and (ii) of the preceding section, in the formulæ (iii) and (iv) of § 2, after division by a, viz. in

$$\tfrac{1}{2} \frac{\cos (x - \frac{1}{2}) a}{\cos \frac{1}{2} a} = A_1'(x) - \frac{a^2}{2!} A_3'(x) + \frac{a^4}{4!} A_5'(x) - \&c.,$$

$$\tfrac{1}{2} \frac{\sin (x - \frac{1}{2}) a}{\sin \frac{1}{2} a} = a A_2'(x) - \frac{a^3}{3!} A_4'(x) + \frac{a^5}{5!} A_6'(x) - \&c.,$$

we find

$$\text{(i)} \quad \frac{\cos (x - \frac{1}{2}) a}{\cos \frac{1}{2} a} = 4 \sin \pi x \int_0^\infty \frac{\cosh \pi t \cosh at}{\cosh 2\pi t - \cos 2\pi x} \, dt,$$

$$\text{(ii)} \quad \frac{\sin (x - \frac{1}{2}) a}{\cos \frac{1}{2} a} = -4 \cos \pi x \int_0^\infty \frac{\sinh \pi t \sinh at}{\cosh 2\pi t - \cos 2\pi x} \, dt.$$

In (i) x must lie between the limits 0 and 1; and in (ii) x must not exceed these limits. The quantity a must be less than π, as otherwise the integrals are not finite.

Putting the integrals on the left-hand side of the equations, these formulæ become

$$\text{(iii)} \quad \int_0^\infty \frac{\cosh \pi t \cosh at}{\cosh 2\pi t - \cos 2\pi x} \, dt = \tfrac{1}{4} \frac{\cos (x - \frac{1}{2}) a}{\cos \frac{1}{2} a \sin \pi x},$$

$$\text{(iv)} \quad \int_0^\infty \frac{\sinh \pi t \sinh at}{\cosh 2\pi t - \cos 2\pi x} \, dt = -\tfrac{1}{4} \frac{\sin (x - \frac{1}{2}) a}{\cos \frac{1}{2} a \cos \pi x}.$$

§ 6. If we substitute ai in the expansion-formulæ, (iii) and (iv) of § 2, the corresponding integral-evaluations are:

$$\text{(i)} \quad \int_0^\infty \frac{\cosh \pi t \cos at}{\cosh 2\pi t - \cos 2\pi x} \, dt = \tfrac{1}{4} \frac{\cosh (x - \frac{1}{2}) a}{\cosh \frac{1}{2} a \sin \pi x},$$

$$\text{(ii)} \quad \int_0^\infty \frac{\sinh \pi t \sin at}{\cosh 2\pi t - \cos 2\pi x}\, dt = -\tfrac{1}{4} \frac{\sinh (x - \frac{1}{2})\, a}{\cosh \frac{1}{2} a \cos \pi x}.$$

In both formulæ the limits of x are the same as before. The quantity a may now have any real value.

§ 7. Taking the integrals in the first form (§ 5), and substituting z for $x - \frac{1}{2}$, they become

$$\text{(i)} \quad \int_0^\infty \frac{\cosh \pi t \cosh at}{\cosh 2\pi t + \cos 2\pi z}\, dt = \tfrac{1}{4} \frac{\cos za}{\cos \frac{1}{2} a \cos \pi z},$$

$$\text{(ii)} \quad \int_0^\infty \frac{\sinh \pi t \sinh at}{\cosh 2\pi t + \cos 2\pi z}\, dt = \tfrac{1}{4} \frac{\sin za}{\cos \frac{1}{2} a \sin \pi z}.$$

In (i) z must lie between $\pm \frac{1}{2}$, and in (ii) z must not exceed these limits. The quantity a must be less than π.

Expressions for $A_{2n+1}(x)$ and $A_{2n}(x)$ as definite integrals, §§ 8–9.

§ 8. The formulæ relating to $A_n(x)$ which correspond to the A'-formulæ of § 2 are (*Quarterly Journal*, §§ 37, 38, pp. 24–25)

$$\text{(i)} \; \sin 2\pi x + \frac{\sin 4\pi x}{2^{2n+1}} + \frac{\sin 6\pi x}{3^{2n+1}} + \&\text{c.} = (-1)^{n+1} \frac{2^{2n} \pi^{2n+1}}{(2n)!} A_{2n+1}(x),$$

$$\text{(ii)} \; \cos 2\pi x + \frac{\cos 4\pi x}{2^{2n}} + \frac{\cos 6\pi x}{3^{2n}} + \&\text{c.} = (-1)^{n-1} \frac{2^{2n-1} \pi^{2n}}{(2n-1)!} A_{2n}(x),$$

$$\text{(iii)} \; \tfrac{1}{2} a \frac{\sin (x - \frac{1}{2})\, a}{\sin \frac{1}{2} a} = a A_1(x) - \frac{a^3}{2!} A_3(x) + \frac{a^5}{4!} A_5(x) - \&\text{c.},$$

$$\text{(iv)} \; \tfrac{1}{2} a \frac{\cos (x - \frac{1}{2})\, a}{\sin \frac{1}{2} a} = 1 - a^2 A_2(x) + \frac{a^4}{3!} A_4(x) - \frac{a^6}{5!} A_6(x) + \&\text{c.},$$

$$\text{(v)} \; \frac{a e^{ax}}{e^a - 1} = 1 + a A_1(x) + a^2 A_2(x) + \frac{a^3}{2!} A_3(x) + \&\text{c.}$$

In (i) and (ii) x must not exceed the limits 0 and 1; but when $n = 0$, (i) does not hold good at these limits. In the other formulæ there is no restriction.

§ 9. Proceeding as in §§ 3 and 4, we find that

$$\text{(i)}\ A_{2n+1}(x) = (-1)^{n+1} \sin 2\pi x \int_0^\infty \frac{t^{2n}\,dt}{\cosh 2\pi t - \cos 2\pi x}$$

$$\text{(ii)}\ A_{2n}(x) = (-1)^{n-1} \int_0^\infty \frac{(\cos 2\pi x - e^{-2\pi t})\, t^{2n-1}\,dt}{\cosh 2\pi t - \cos 2\pi x};$$

or, putting the integrals on the left-hand side of the equations,

$$\text{(iii)} \int_0^\infty \frac{t^{2n}\,dt}{\cosh 2\pi t - \cos 2\pi x} = (-1)^{n+1} \frac{A_{2n+1}(x)}{\sin 2\pi x},$$

$$\text{(iv)} \int_0^\infty \frac{(\cos 2\pi x - e^{-2\pi t})\, t^{2n-1}\,dt}{\cosh 2\pi t - \cos 2\pi x} = (-1)^{n-1} A_{2n}(x).$$

In formulæ (i)–(iv) x must not exceed the limits 0 and 1. Formulæ (i) and (iii) do not hold good at these limits in the special case of $n = 0$.

Evaluation of general integrals, § 10–12.

§ 10. Substituting in (iii) of § 8 the values of $A_1(x)$, $A_3(x)$, &c. given by (i) of § 9, we find

$$\tfrac{1}{2} \frac{\sin (x - \frac{1}{2}) a}{\sin \frac{1}{2} a} = -\sin 2\pi x \int_0^\infty \frac{\cosh at\,dt}{\cosh 2\pi t - \cos 2\pi x}.$$

From (iv) of § 9, we have

$$\tfrac{1}{2} \frac{\cos (x - \frac{1}{2}) a}{\sin \frac{1}{2} a} - \frac{1}{a} = -aA_2(x) + \frac{a^3}{3\,!} A_4(x) - \frac{a^5}{5\,!} A_6(x) + \&\text{c.},$$

$$= -\int_0^\infty \frac{(\cos 2\pi x - e^{-2\pi t}) \sinh at}{\cosh 2\pi t - \cos 2\pi x}\, dt.$$

Thus we have

$$\text{(i)} \int_0^\infty \frac{\cosh at\,dt}{\cosh 2\pi t - \cos 2\pi x} = -\tfrac{1}{2} \frac{\sin (x - \frac{1}{2}) a}{\sin \frac{1}{2} a \sin 2\pi x},$$

$$\text{(ii)} \int_0^\infty \frac{(\cos 2\pi x - e^{-2\pi t}) \sinh at}{\cosh 2\pi t - \cos 2\pi x}\, dt = \frac{1}{a} - \tfrac{1}{2} \frac{\cos (x - \frac{1}{2}) a}{\sin \frac{1}{2} a}.$$

In (i) x must lie between 0 and 1. In (ii) x must not exceed these limits. In both formulæ a must be less than 2π.

§ 11. Putting $z = x - \frac{1}{2}$, as in § 7, the integrals become

$$\text{(i)} \int_0^\infty \frac{\cosh at\,dt}{\cosh 2\pi t + \cos 2\pi z} = \frac{1}{2}\,\frac{\sin za}{\sin \frac{1}{2}a \sin 2\pi z},$$

$$\text{(ii)} \int_0^\infty \frac{(\cos 2\pi z + e^{-2\pi t}) \sinh at}{\cosh 2\pi t + \cos 2\pi z}\,dt = -\frac{1}{a} + \frac{1}{2}\,\frac{\cos za}{\sin \frac{1}{2}a}.$$

In (i) z must lie between $\pm \frac{1}{2}$, and in (ii) it must not exceed these limits.

§ 12. It may be remarked that (i) may be derived by addition or subtraction from (i) and (ii) of § 7. For by adding these formulæ, we find

$$\int_0^\infty \frac{\cosh (\pi + a)\,t}{\cosh 2\pi t + \cos 2\pi z}\,dt = \frac{1}{4}\,\frac{\sin (\pi + a)\,z}{\cos \frac{1}{2}a \sin \pi z \cos \pi z},$$

when, replacing $\pi + a$ by a,

$$\int_0^\infty \frac{\cosh at\,dt}{\cosh 2\pi t + \cos 2\pi z} = \frac{1}{2}\,\frac{\sin az}{\cos \frac{1}{2}a \sin 2\pi z}.$$

The four general integrals, §§ 13–16.

§ 13. Collecting the formulæ obtained in §§ 7 and 11, and replacing a by ai, we have the four results:

$$\text{(i)} \int_0^\infty \frac{\cosh \pi t}{\cosh 2\pi t + \cos 2\pi z} \cos at\,dt = \frac{1}{4}\,\frac{\cosh za}{\cosh \frac{1}{2}a \cos \pi z},$$

$$\text{(ii)} \int_0^\infty \frac{\sinh \pi t}{\cosh 2\pi t + \cos 2\pi z} \sin at\,dt = \frac{1}{4}\,\frac{\sinh za}{\cosh \frac{1}{2}a \sin \pi z},$$

$$\text{(iii)} \int_0^\infty \frac{dt}{\cosh 2\pi t + \cos 2\pi z} \cos at\,dt = \frac{1}{2}\,\frac{\sinh za}{\sinh \frac{1}{2}a \sin 2\pi z},$$

$$\text{(iv)} \int_0^\infty \frac{\cos 2\pi z + e^{-2\pi t}}{\cosh 2\pi t + \cos 2\pi z} \sin at\,dt = \frac{1}{a} - \frac{1}{2}\,\frac{\cosh za}{\sinh \frac{1}{2}a}.$$

In (i) and (iii) z must lie between $\pm \frac{1}{2}$; in (ii) and (iv) it must not exceed these limits. The value of a is unrestricted.

§ 14. These formulæ may be readily derived from the following results, which are due to Poisson:

$$\int_0^\infty \frac{\cosh px}{\cosh qx}\cos ax\,dx = \frac{\pi}{q}\,\frac{\cosh\dfrac{a\pi}{2q}\cos\dfrac{p\pi}{2q}}{\cosh\dfrac{a\pi}{q}+\cos\dfrac{p\pi}{q}},$$

$$\int_0^\infty \frac{\sinh px}{\cosh qx}\sin ax\,dx = \frac{\pi}{q}\,\frac{\sinh\dfrac{a\pi}{2q}\sin\dfrac{p\pi}{2q}}{\cosh\dfrac{a\pi}{q}+\cos\dfrac{p\pi}{q}},$$

$$\int_0^\infty \frac{\sinh px}{\sinh qx}\cos ax\,dx = \frac{\pi}{2q}\,\frac{\sin\dfrac{p\pi}{q}}{\cosh\dfrac{a\pi}{q}+\cos\dfrac{p\pi}{q}},$$

$$\int_0^\infty \frac{\cosh px}{\sinh qx}\sin ax\,dx = \frac{\pi}{2q}\,\frac{\sinh\dfrac{a\pi}{q}}{\cosh\dfrac{a\pi}{q}+\cos\dfrac{p\pi}{q}}.$$

§ 15. To obtain the results of § 13, we make use of the known reciprocity of f and ϕ in each of the formulæ

$$f(a) = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \phi(x)\cos ax\,dx,$$

$$f(a) = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\infty \phi(x)\sin ax\,dx.$$

For example, taking Poisson's first result, we have

$$\phi(x) = \frac{\cosh px}{\cosh qx},$$

$$\sqrt{\left(\frac{\pi}{2}\right)}f(a) = \frac{\pi}{q}\,\frac{\cosh\dfrac{a\pi}{2q}\cos\dfrac{p\pi}{2q}}{\cosh\dfrac{a\pi}{q}+\cos\dfrac{p\pi}{q}};$$

whence, interchanging ϕ and f,

$$\frac{2}{\pi}\int_0^\infty \frac{\pi}{q}\,\frac{\sinh\frac{\pi x}{2q}\cos\frac{p\pi}{2q}}{\cosh\frac{\pi x}{q}+\cos\frac{p\pi}{q}}\cos ax\,dx=\frac{\cosh pa}{\cosh qa}.$$

Putting $x=2qt$, $a=\frac{a}{2q}$, $\frac{p}{q}=2z$, this equation becomes

$$4\int_0^\infty \frac{\sinh\pi t\cos\pi z}{\cosh 2\pi t+\cos 2\pi z}\cos at\,dt=\frac{\cosh za}{\cosh\frac{1}{2}a},$$

which is equation (i) of § 13.

§ 16. In a similar manner we may derive (ii) and (iii) from Poisson's second and third results, but by applying the same process to Poisson's fourth result, we obtain, in place of (iv), the equation,

$$(\alpha)\quad \int_0^\infty \frac{\sinh 2\pi t}{\cosh 2\pi t+\cos 2\pi z}\sin at\,dt=\frac{1}{2}\,\frac{\cosh 2a}{\sinh\frac{1}{2}a}.$$

The integral on the left-hand side of this equation is indeterminate in value; but we can derive (iv) from it, if we may put

$$(\beta)\quad \int_0^\infty \sin at\,dt=\frac{1}{a},$$

for
$$\frac{\sinh 2\pi t}{\cosh 2\pi t+\cos 2\pi z}=1-\frac{\cos 2\pi z+e^{-2\pi t}}{\cosh 2\pi t+\cos 2\pi x}.$$

The formula (β) is true if we regard $\int_0^\infty \sin at\,dt$ as the limiting value of $\int_0^\infty e^{-kt}\sin at\,dt$, when $k=0$*; and similarly the equation (α) is true if we regard the integral as the limit, when $k=0$, of

$$\int_0^\infty e^{-kt}\frac{\sinh 2\pi t}{\cosh 2\pi t+\cos 2\pi z}\sin at\,dt.$$

---

* *Messenger of Mathematics*, First Series, Vol. v., p. 236.

§ 17. It may be remarked that the formula (iv) of § 13 may also be written in the form

$$\text{(iv)}\quad \int_0^\infty \left(1 - \frac{\sinh 2\pi t}{\cosh 2\pi t + \cos 2\pi z}\right) \sin at\, dt = \frac{1}{a} - \frac{1}{2}\frac{\cosh za}{\sinh \frac{1}{2}a}.$$

Particular cases of the general integrals, §§ 18–27.

§ 18. I now proceed to consider the particular cases of the general integrals in § 13 which may be obtained by assigning special values to z.

Putting $z=0$ in (i) and $z=\frac{1}{2}$ in (iv), we find

$$\text{(i)}\quad \int_0^\infty \frac{\cos at}{\cosh \pi t}\, dt = \frac{1}{2}\frac{1}{\cosh \frac{1}{2}a},$$

$$\text{(ii)}\quad \int_0^\infty \frac{\sin at}{\sinh \pi t}\, dt = \frac{1}{2}\tanh \tfrac{1}{2}a.$$

§ 19. Putting $z=\frac{1}{4}$, the formulæ (i) and (ii) give

$$\text{(i)}\quad \int_0^\infty \frac{\cosh \pi t}{\cosh 2\pi t} \cos at\, dt = \frac{1}{2\sqrt{2}}\frac{\cosh \frac{1}{4}a}{\cosh \frac{1}{2}a},$$

$$\text{(ii)}\quad \int_0^\infty \frac{\sinh \pi t}{\cosh 2\pi t} \sin at\, dt = \frac{1}{2\sqrt{2}}\frac{\sinh \frac{1}{4}a}{\cosh \frac{1}{2}a}.$$

Formula (iii) merely reproduces (i) of § 18.

§ 20. Putting $z=\frac{1}{3}$, (i), (ii), (iii) give

$$\text{(i)}\quad \int_0^\infty \frac{\cosh \pi t}{2\cosh 2\pi t - 1} \cos at\, dt = \frac{1}{4}\frac{\cosh \frac{1}{3}a}{\cosh \frac{1}{2}a},$$

$$\text{(ii)}\quad \int_0^\infty \frac{\sinh \pi t}{2\cosh 2\pi t - 1} \sin at\, dt = \frac{1}{4\sqrt{3}}\frac{\sinh \frac{1}{3}a}{\cosh \frac{1}{2}a},$$

$$\text{(iii)}\quad \int_0^\infty \frac{\cos at\, dt}{2\cosh 2\pi t - 1} = \frac{1}{2\sqrt{3}}\frac{\sinh \frac{1}{3}a}{\sinh \frac{1}{2}a}.$$

§ 21. Using exponentials instead of hyperbolic functions under the integral sign, these formulæ may be written

$$\text{(i)}\quad \int_0^\infty \frac{e^{\pi t} + e^{-\pi t}}{e^{2\pi t} + e^{-2\pi t} - 1} \cos at\, dt = \frac{1}{2}\frac{\cosh \frac{1}{3}a}{\cosh \frac{1}{2}a},$$

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$$\text{(ii)} \quad \int_0^\infty \frac{e^{\pi t} - e^{-\pi t}}{e^{2\pi t} + e^{-2\pi t} - 1} \sin at \, dt = \frac{1}{2\sqrt{3}} \, \frac{\sinh \frac{1}{3}a}{\cosh \frac{1}{2}a},$$

$$\text{(iii)} \quad \int_0^\infty \frac{\cos at \, dt}{e^{2\pi t} + e^{-2\pi t} - 1} = \frac{1}{2\sqrt{3}} \, \frac{\sinh \frac{1}{3}a}{\sinh \frac{1}{2}a}.$$

§ 22. Since

$$\cosh x (2 \cosh 2x - 1) = \cosh 3x,$$

we may write these integrals also in the form

$$\text{(i)} \quad \int_0^\infty \frac{\cosh^2 \pi t}{\cosh 3\pi t} \cos at \, dt = \frac{1}{4} \, \frac{\cosh \frac{1}{3}a}{\cosh \frac{1}{2}a},$$

$$\text{(ii)} \quad \int_0^\infty \frac{\sinh 2\pi t}{\cosh 3\pi t} \sin at \, dt = \frac{1}{2\sqrt{3}} \, \frac{\sinh \frac{1}{3}a}{\cosh \frac{1}{2}a},$$

$$\text{(iii)} \quad \int_0^\infty \frac{\cosh \pi t}{\cosh 3\pi t} \cos at \, dt = \frac{1}{2\sqrt{3}} \, \frac{\sinh \frac{1}{3}a}{\sinh \frac{1}{2}a}.$$

§ 23. The corresponding formulæ obtained by putting $x = \frac{1}{6}$ are

$$\text{(i)} \quad \int_0^\infty \frac{\cosh \pi t}{2 \cosh 2\pi t + 1} \cos at \, dt = \frac{1}{4\sqrt{3}} \, \frac{\cosh \frac{1}{6}a}{\cosh \frac{1}{2}a},$$

$$\text{(ii)} \quad \int_0^\infty \frac{\sinh \pi t}{2 \cosh 2\pi t + 1} \sin at \, dt = \frac{1}{4} \, \frac{\sinh \frac{1}{6}a}{\cosh \frac{1}{2}a},$$

$$\text{(iii)} \quad \int_0^\infty \frac{\cos at \, dt}{2 \cosh 2\pi t + 1} = \frac{1}{2\sqrt{3}} \, \frac{\sinh \frac{1}{6}a}{\sinh \frac{1}{2}a},$$

or, using exponentials on the left-hand side,

$$\text{(i)} \quad \int_0^\infty \frac{e^{\pi t} + e^{-\pi t}}{e^{2\pi t} + e^{-2\pi t} + 1} \cos at \, dt = \frac{1}{2\sqrt{3}} \, \frac{\cosh \frac{1}{6}a}{\cosh \frac{1}{2}a},$$

$$\text{(ii)} \quad \int_0^\infty \frac{e^{\pi t} - e^{-\pi t}}{e^{2\pi t} + e^{-2\pi t} + 1} \sin at \, dt = \frac{1}{2} \, \frac{\sinh \frac{1}{6}a}{\cosh \frac{1}{2}a},$$

$$\text{(iii)} \quad \int_0^\infty \frac{\cos at \, dt}{e^{2\pi t} + e^{-2\pi t} + 1} = \frac{1}{2\sqrt{3}} \, \frac{\sinh \frac{1}{6}a}{\sinh \frac{1}{2}a}.$$

§ 24. Since

$$\sinh x(2\cosh 2x+1)=\sinh 3x,$$

these formulæ may also be written

$$\text{(i)}\quad \int_0^\infty \frac{\sinh 2\pi t}{\sinh 3\pi t}\cos at\,dt=\frac{1}{2\sqrt{3}}\,\frac{\cosh\frac{1}{6}a}{\cosh\frac{1}{2}a},$$

$$\text{(ii)}\quad \int_0^\infty \frac{\sinh^2\pi t}{\sinh 3\pi t}\sin at\,dt=\tfrac{1}{4}\,\frac{\sinh\frac{1}{6}a}{\cosh\frac{1}{2}a},$$

$$\text{(iii)}\quad \int_0^\infty \frac{\sinh \pi t}{\sinh 3\pi t}\cos at\,dt=\frac{1}{2\sqrt{3}}\,\frac{\sinh\frac{1}{6}a}{\sinh\frac{1}{2}a}.$$

§ 25. The most convenient form of equation (iv) of § 13 is probably that given in § 17. Putting $z=0, \frac{1}{2}, \frac{1}{3}, \frac{1}{6}$ in this formula, we find

$$\int_0^\infty (1-\tanh\pi t)\sin at\,dt=\frac{1}{a}-\tfrac{1}{2}\,\frac{1}{\sinh\frac{1}{2}a},$$

$$\int_0^\infty (1-\coth\pi t)\sin at\,dt=\frac{1}{a}-\tfrac{1}{2}\coth\tfrac{1}{2}a,$$

$$\int_0^\infty \left(1-\frac{2\sinh 2\pi t}{2\cosh 2\pi t-1}\right)\sin at\,dt=\frac{1}{a}-\tfrac{1}{2}\,\frac{\cosh\frac{1}{3}a}{\sinh\frac{1}{2}a},$$

$$\int_0^\infty \left(1-\frac{2\sinh 2\pi t}{2\cosh 2\pi t+1}\right)\sin at\,dt=\frac{1}{a}-\tfrac{1}{2}\,\frac{\cosh\frac{1}{6}a}{\sinh\frac{1}{2}a}.$$

By putting $z=\frac{1}{4}$ we merely reproduce the first result.

The last two formulæ may be written in the form

$$\int_0^\infty \left(1-\frac{2\cosh\pi t\sinh 2\pi t}{\cosh 3\pi t}\right)\sin at\,dt=\frac{1}{a}-\tfrac{1}{2}\,\frac{\cosh\frac{1}{3}a}{\sinh\frac{1}{2}a},$$

$$\int_0^\infty \left(1-\frac{2\sinh\pi t\sinh 2\pi t}{\sinh 3\pi t}\right)\sin at\,dt=\frac{1}{a}-\tfrac{1}{2}\,\frac{\cosh\frac{1}{6}a}{\sinh\frac{1}{2}a},$$

and also in the form

$$\int_0^\infty \left(1-\tanh 3\pi t-\frac{\sinh\pi t}{\cosh 3\pi t}\right)\sin at\,dt=\frac{1}{a}-\tfrac{1}{2}\,\frac{\cosh\frac{1}{3}a}{\sinh\frac{1}{2}a},$$

$$\int_0^\infty \left(1-\coth 3\pi t+\frac{\cosh\pi t}{\sinh 3\pi t}\right)\sin at\,dt=\frac{1}{a}-\tfrac{1}{2}\,\frac{\cosh\frac{1}{6}a}{\sinh\frac{1}{2}a}.$$

§ 26. None of these results can be regarded as actually new, for they are all derivable from Poisson's formulæ of § 14. These latter formulæ also enable to complete the groups, the resulting system of equations being:

$$\int_0^\infty \frac{\cosh \pi t}{\cosh 2\pi t} \cos at\, dt = \frac{1}{2\sqrt{2}} \frac{\cosh \frac{1}{4}a}{\cosh \frac{1}{2}a},$$

$$\int_0^\infty \frac{\sinh \pi t}{\cosh 2\pi t} \sin at\, dt = \frac{1}{2\sqrt{2}} \frac{\sinh \frac{1}{4}a}{\cosh \frac{1}{2}a},$$

$$\int_0^\infty \frac{\sinh \pi t}{\sinh 2\pi t} \cos at\, dt = \frac{1}{2} \int_0^\infty \frac{\cos at\, dt}{\cosh \pi t} = \frac{1}{4} \frac{1}{\cosh \frac{1}{2}a},$$

$$\int_0^\infty \frac{\cosh \pi t}{\cosh 2\pi t} \sin at\, dt = \frac{1}{2} \int_0^\infty \frac{\sin at\, dt}{\sinh \pi t} = \frac{1}{4} \tanh \tfrac{1}{2}a,$$

$$\int_0^\infty \frac{\cosh \pi t}{\cosh 3\pi t} \cos at\, dt = \frac{1}{2\sqrt{3}} \frac{\sinh \frac{1}{3}a}{\sinh \frac{1}{2}a},$$

$$\int_0^\infty \frac{\sinh \pi t}{\cosh 3\pi t} \sin at\, dt = \frac{1}{3} \frac{\sinh^2 \frac{1}{6}a}{\sinh \frac{1}{3}a},$$

$$\int_0^\infty \frac{\sinh \pi t}{\sinh 3\pi t} \cos at\, dt = \frac{1}{2\sqrt{3}} \frac{\sinh \frac{1}{6}a}{\sinh \frac{1}{2}a},$$

$$\int_0^\infty \frac{\cosh \pi t}{\cosh 3\pi t} \sin at\, dt = \frac{1}{3} \frac{\sinh \frac{1}{6}a \sinh \frac{1}{3}a}{\sinh \frac{1}{2}a},$$

$$\int_0^\infty \frac{\cosh 2\pi t}{\cosh 3\pi t} \cos at\, dt = \frac{1}{3} \frac{\cosh^2 \frac{1}{6}a}{\cosh \frac{1}{2}a},$$

$$\int_0^\infty \frac{\sinh 2\pi t}{\cosh 3\pi t} \sin at\, dt = \frac{1}{2\sqrt{3}} \frac{\sinh \frac{1}{3}a}{\cosh \frac{1}{2}a},$$

$$\int_0^\infty \frac{\sinh 2\pi t}{\sinh 3\pi t} \cos at\, dt = \frac{1}{2\sqrt{3}} \frac{\cosh \frac{1}{6}a}{\cosh \frac{1}{2}a},$$

$$\int_0^\infty \frac{\cosh 2\pi t}{\sinh 3\pi t} \sin at\, dt = \frac{1}{3} \frac{\cosh \frac{1}{6}a \sinh \frac{1}{3}a}{\cosh \frac{1}{2}a},$$

$$\int_0^\infty \frac{\cosh^2 \pi t}{\cosh 3\pi t} \cos at\, dt = \frac{1}{2} \frac{\cosh \frac{1}{3}a}{\cosh \frac{1}{2}a},$$

$$\int_0^\infty \frac{\sinh^2 \pi t}{\sinh 3\pi t} \sin at\, dt = \frac{1}{2} \frac{\sinh \frac{1}{6}a}{\cosh \frac{1}{2}a}.$$

We also have, from § 25,

$$\int_0^\infty (1-\tanh \pi t) \sin at \, dt = \frac{1}{a} - \tfrac{1}{2}\frac{1}{\sinh \frac{1}{2}a},$$

$$\int_0^\infty (1-\coth \pi t) \sin at \, dt = \frac{1}{a} - \tfrac{1}{2}\coth \tfrac{1}{2}a,$$

$$\int_0^\infty \left(1-\frac{2\cosh \pi t \sinh 2\pi t}{\cosh 3\pi t}\right) \sin at \, dt = \frac{1}{a} - \tfrac{1}{2}\frac{\cosh \frac{1}{3}a}{\sinh \frac{1}{2}a},$$

$$\int_0^\infty \left(1-\frac{2\sinh \pi t \sinh 2\pi t}{\sinh 3\pi t}\right) \sin at \, dt = \frac{1}{a} - \tfrac{1}{2}\frac{\cosh \frac{1}{6}a}{\sinh \frac{1}{2}a}.$$

§ 27. If we transform, say, the first formula by putting $\pi t = x$ and replace a by πa, it becomes

$$\int_0^\infty \frac{\cosh x}{\cosh 2x} \cos at \, dt = \frac{\pi}{2\sqrt{2}} \, \frac{\cosh \frac{1}{4}\pi a}{\cosh \frac{1}{2}\pi a}.$$

Thus we may in all the results suppose the π to be transferred from the hyperbolic functions under the integral sign to the hyperbolic functions on the right-hand side of the equation, if at the same time we introduce the factor π on the right-hand side.

Expansion-formulæ, §§ 28–32.

§ 28. It is well-known that, B_1, B_2, B_3, ..., being the Bernoullian numbers,

$$\frac{a}{\sinh a} = 1 - \frac{(2^2-2)B_1}{2!}a^2 + \frac{(2^4-2)B_2}{4!}a^4 - \frac{(2^6-2)B_3}{6!}a^6 + \&c.,$$

$$\tanh a = \frac{2^2(2^2-1)B_1}{2!}a - \frac{2^4(2^4-1)B_2}{4!}a^3 + \frac{2^6(2^6-1)B_3}{6!}a^5 - \&c.,$$

$$a \coth a = 1 + \frac{2^2 B_1}{2!}a^2 - \frac{2^4 B_2}{4!}a^4 + \frac{2^6 B_3}{6!}a^6 - \&c.,$$

and that, E_1, E_2, E_3, ..., being the Eulerian numbers,

$$\frac{1}{\cosh a} = 1 - \frac{E_1}{2!}a^2 + \frac{E_2}{4!}a^4 - \frac{E_3}{6!}a^6 + \&c.$$

§ 29. In the paper in the *Quarterly Journal* referred to in § 1 use was made of the numbers I_n, H_n, J_n defined by the equations

$$\frac{1}{e^a+e^{-a}+1}=\frac{1}{2\cosh a+1}=\frac{\sinh\frac{1}{2}a}{\sinh\frac{3}{2}a}$$

$$=\tfrac{2}{3}\left\{I_0-\frac{I_1}{2!}a^2+\frac{I_2}{4!}a^4-\frac{I_3}{6!}a^6+\&c.\right\},\ (\S\ 57,\ \text{p. } 35),$$

$$\frac{1}{e^a+e^{-a}-1}=\frac{1}{2\cosh a-1}=\frac{\cosh\frac{1}{2}a}{\cosh\frac{3}{2}a}$$

$$=\tfrac{2}{3}\left\{H_0-\frac{H_1}{2!}a^2+\frac{H_2}{4!}a^4-\frac{H_3}{6!}a^6+\&c.\right\},\ (\S\ 84,\ \text{p. } 48),$$

$$\frac{e^a+e^{-a}}{e^{2a}+e^{-2a}+1}=\frac{2\cosh a}{2\cosh 2a+1}=\frac{\sinh 2a}{\sinh 3a}$$

$$=\tfrac{1}{3}\left\{J_0-\frac{J_1}{2!}a^2+\frac{J_2}{4!}a^4-\frac{J_3}{6!}a^6+\&c.\right\},\ (\S\ 75,\ \text{p. } 44).$$

The references are to the sections and pages in the *Quarterly Journal.**

§ 30. The coefficients I_n, H_n, J_n are connected by the relations

$$H_n=(2^{2n+1}+1)\,I_n,\quad (\S\ 84,\ \text{p. } 49),$$
$$J_n=(2^{2n+1}+2)\,I_n,\quad (\S\ 75,\ \text{p. } 44);$$

so that

$$I_n+H_n=J_n.$$

The first six values of the I-, H-, and J-numbers are

$$\begin{array}{lll}
I_0=\frac{1}{2}, & H_0=\frac{3}{2}, & J_0=2,\\
I_1=\frac{1}{3}, & H_1=3, & J_1=\frac{10}{3},\\
I_2=1, & H_2=33, & J_2=34,\\
I_3=7, & H_3=903, & J_3=910,\\
I_4=\frac{809}{9}, & H_4=46113, & J_4=\frac{415826}{9},\\
I_5=1847, & H_5=3784503, & J_5=3786350,
\end{array}$$

(§ 58, p. 56, and § 85, p. 49).

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* Throughout the remainder of this paper the references, following the formulæ, to section and page relate to the paper in the *Quarterly Journal.*

§ 31. The following expansions are also employed in the same paper in connection with special values of the Bernoullian Function:

$$\frac{\cosh a}{\cosh 2a} = P_0 - \frac{P_1}{2!}a^2 + \frac{P_2}{4!}a^4 - \&c., \quad (§\ 109,\ p.\ 60),$$

$$\frac{\sinh a}{\cosh 2a} = Q_1 a - \frac{Q_2}{3!}a^3 + \frac{Q_3}{5!}a^5 - \&c., \quad (§\ 119,\ p.\ 64),$$

$$\frac{1}{2}\frac{e^a + e^{-a}}{e^{2a} + e^{-2a} - 1} = \frac{\cosh a}{2\cosh 2a - 1} = \frac{\cosh^2 a}{\cosh 3a}$$

$$= R_0 - \frac{R_1}{2!}a^2 + \frac{R_2}{4!}a^4 - \&c., \quad (§\ 137,\ p.\ 72),$$

$$\frac{1}{2}\frac{e^a - e^{-a}}{e^{2a} + e^{-2a} - 1} = \frac{\sinh a}{2\cosh 2a - 1} = \frac{1}{2}\frac{\sinh 2a}{\cosh 3a}$$

$$= T_1 a - \frac{T_2}{3!}a^3 + \frac{T_3}{5!}a^5 - \&c., \quad (§\ 143,\ p.\ 75).$$

§ 32. The R-numbers are connected with the Eulerian numbers by the simple relation

$$R_n = \frac{3^{2n+1} + 1}{4} E_n, \quad (§\ 134\ p.\ 71).$$

The first six values of the P-, Q-, R-, T-numbers are:

$P_1 = 3$, $P_2 = 57$, $P_3 = 2763$, $P_4 = 250737$, $P_5 = 36581523$,
(§ 115, p. 63),

$Q_1 = 1$, $Q_2 = 11$, $Q_3 = 361$, $Q_4 = 24611$, $Q_5 = 2873041$,
(§ 123, p. 66),

$R_1 = 7$, $R_2 = 305$, $R_3 = 33367$, $R_4 = 6815585$, $R_5 = 2237423527$,
(§ 134, p. 71),

$T_1 = 1$, $T_2 = 23$, $T_3 = 1681$, $T_4 = 257543$, $T_5 = 67637281$,
(§ 146, p. 76).

Expressions for Bernoullian and Eulerian numbers as definite integrals, §§ 33–35.

§ 33. From §§ 28 and 26, we have

$$1-\frac{(2^2-2)B_1}{2!}a^2+\frac{(2^4-2)B_2}{4!}a^4-\frac{(2^6-2)B_3}{6!}a^6+\&c.$$

$$=\frac{a}{\sinh a}=1-2a\int_0^\infty(1-\tanh\pi t)\sin 2at\,dt;$$

whence, equating the coefficients of a^{2n},

$$(2^{2n}-2)B_n=2^{2n}.2n\int_0^\infty t^{2n-1}(1-\tanh\pi t)\,dt$$

$$=2^{2n}.2n\int_0^\infty\frac{t^{2n-1}.2e^{-\pi t}}{e^{\pi t}+e^{-\pi t}}dt,$$

and therefore

$$\text{(i)}\quad\int_0^\infty\frac{t^{2n-1}dt}{e^{2\pi t}+1}=\frac{(2^{2n-1}-1)B_n}{2^{2n}.2n}=\left(1-\frac{1}{2^{2n-1}}\right)\frac{B_n}{4n}.$$

Similarly, from the equations

$$1+\frac{2^2B_1}{2!}a^2-\frac{2^4B_2}{4!}a^4+\frac{2^6B_3}{6!}a^6-\&c.$$

$$=a\coth a=1-2a\int_0^\infty(1-\coth\pi t)\sin 2at\,dt,$$

we find

$$\text{(ii)}\quad\int_0^\infty\frac{t^{2n-1}dt}{e^{2\pi t}-1}=\frac{B_n}{4n},$$

§ 34. The equations

$$\frac{2^2(2^2-1)B_1}{2!}a-\frac{2^4(2^4-1)B_2}{4!}a^3+\frac{2^6(2^6-1)B_3}{6!}a^5-\&c.$$

$$=\tanh a=2\int_0^\infty\frac{\sin 2at\,dt}{\sinh\pi t},$$

give

$$\int_0^\infty\frac{t^{2n-1}dt}{\sinh\pi t}=\frac{(2^{2n}-1)B_n}{2n};$$

or, expressing the left-hand side in exponentials,

$$\text{(iii)}\quad\int_0^\infty\frac{t^{2n-1}\,dt}{e^{\pi t}-e^{-\pi t}}=(2^{2n}-1)\frac{B_n}{4n}.$$

Similarly, from the equations

$$1-\frac{E_1}{2!}a^2+\frac{E_2}{4!}a^4-\frac{E_3}{6!}a^6+\&c.=\frac{1}{\cosh a}=2\int_0^\infty\frac{\cos 2at\,dt}{\cosh \pi t},$$

we find

$$\int_0^\infty \frac{t^{2n}dt}{\cosh \pi t}=\frac{E_n}{2^{2n+1}},$$

or

$$\text{(iv)}\quad \int_0^\infty \frac{t^{2n}dt}{e^{\pi t}+e^{-\pi t}}=\frac{E_n}{2^{2n+2}}.$$

§ 35. The integrals (i)–(iv), which are well-known, may be proved most easily by expanding the exponential expressions in powers of $e^{-\pi t}$, and integrating the separate terms by means of the formula

$$\int_0^\infty t^n e^{-r\pi t}dt=\frac{n!}{(r\pi)^{n+1}}.$$

The four integrals thus become

$$\text{(i)}\quad \frac{(2n-1)!}{(2\pi)^{2n}}\left\{1-\frac{1}{2^{2n}}+\frac{1}{3^{2n}}-\frac{1}{4^{2n}}+\&c.\right\},$$

$$\text{(ii)}\quad \frac{(2n-1)!}{(2\pi)^{2n}}\left\{1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+\&c.\right\},$$

$$\text{(iii)}\quad \frac{(2n-1)!}{\pi^{2n}}\left\{1+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\frac{1}{7^{2n}}+\&c.\right\},$$

$$\text{(iv)}\quad \frac{(2n)!}{\pi^{2n+1}}\left\{1-\frac{1}{3^{2n}}+\frac{1}{5^{2n}}-\frac{1}{7^{2n}}+\&c.\right\}.$$

and, since

$$1-\frac{1}{2^{2n}}+\frac{1}{3^{2n}}-\frac{1}{4^{2n}}+\&c.=\frac{(2^{2n-1}-1)\pi^{2n}B_n}{(2n)!},$$

$$1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+\&c.=\frac{2^{2n-1}\pi^{2n}B_n}{(2n)!},$$

$$1+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\frac{1}{7^{2n}}+\&c.=\frac{(2^{2n}-1)\pi^{2n}B_n}{2(2n)!},$$

$$1-\frac{1}{3^{2n+1}}+\frac{1}{5^{2n+1}}-\frac{1}{7^{2n+1}}+\&c.=\frac{\pi^{2n+1}E_n}{2^{2n+2}(2n)!},$$

we at once obtain for the four integrals the values assigned to them in §§ 33 and 34.

Expressions for I_n, H_n, J_n as definite integrals, §§ 36–37.

§ 36. From §§ 29 and 26,

$$\tfrac{2}{3}\left\{I_0 - \frac{I_1}{2!}a^2 + \frac{I_2}{4!}a^4 - \frac{I_3}{6!}a^6 + \&c.\right\}$$

$$= \frac{1}{e^a + e^{-a} + 1} = \frac{\sinh \frac{1}{2}a}{\sinh \frac{3}{2}a} = 2\sqrt{3}\int_0^\infty \frac{\sinh \pi t}{\sinh 3\pi t}\cos 3at\, dt;$$

whence, equating coefficients, we find

$$\text{(i)} \quad \int_0^\infty t^{2n}\frac{\sinh \pi t}{\sinh 3\pi t}\, dt = \int_0^\infty \frac{t^{2n}\, dt}{e^{2\pi t} + e^{-2\pi t} + 1} = \frac{I_n}{3^{2n+\frac{3}{2}}}.$$

Similarly, from the equations,

$$\tfrac{2}{3}\left\{H_0 - \frac{H_1}{2!}a^2 + \frac{H_2}{4!}a^4 - \frac{H_3}{6!}a^6 + \&c.\right\}$$

$$= \frac{1}{e^a + e^{-a} - 1} = \frac{\cosh \frac{1}{2}a}{\cosh \frac{3}{2}a} = 2\sqrt{3}\int_0^\infty \frac{\sinh 2\pi t}{\sinh 3\pi t}\cos 3at\, dt,$$

we find

$$\text{(ii)} \quad \int_0^\infty t^{2n}\frac{\sinh 2\pi t}{\sinh 3\pi t}\, dt = \int_0^\infty t^{2n}\frac{e^{\pi t} + e^{-\pi t}}{e^{2\pi t} + e^{-2\pi t} + 1}\, dt = \frac{H_n}{3^{2n+\frac{3}{2}}},$$

and, from the equations

$$\tfrac{1}{3}\left\{J_0 - \frac{J_1}{2!}a^2 + \frac{J_2}{4!}a^4 - \frac{J_3}{6!}a^6 + \&c.\right\}$$

$$= \frac{e^a + e^{-a}}{e^{2a} + e^{-2a} + 1} = \frac{\sinh 2a}{\sinh 3a} = 2\sqrt{3}\int_0^\infty \frac{\cosh \pi t}{\cosh 3\pi t}\cos 6at\, dt,$$

we find

$$\text{(iii)} \quad \int_0^\infty t^{2n}\frac{\cosh \pi t}{\cosh 3\pi t}\, dt = \int_0^\infty \frac{t^{2n}\, dt}{e^{2\pi t} + e^{-2\pi t} - 1} = \frac{J_n}{6^{2n+1}\sqrt{3}}.$$

The formulæ (i) and (iii) may also be written

$$\int_0^\infty \frac{t^{2n}\, dt}{e^{\pi t} + e^{-\pi t} + 1} = \left(\tfrac{2}{3}\right)^{2n+1}\frac{I_n}{\sqrt{3}},$$

$$\int_0^\infty \frac{t^{2n}\, dt}{e^{\pi t} + e^{-\pi t} - 1} = \left(\tfrac{1}{3}\right)^{2n+1}\frac{J_n}{\sqrt{3}}.$$

§ 37. The values of the integrals (i), (ii), (iii) may also be obtained directly by expansion as in § 35; for

$$\frac{\sinh x}{\sinh 3x} = e^{-2x} - e^{-4x} + e^{-8x} - e^{-10x} + e^{-14x} - \&c.,$$

$$\frac{\sinh 2x}{\sinh 3x} = e^{-x} - e^{-5x} + e^{-7x} - e^{-11x} + e^{-13x} - \&c.,$$

$$\frac{\cosh x}{\cosh 3x} = e^{-2x} + e^{-4x} - e^{-8x} - e^{-10x} + e^{-14x} + \&c.,$$

so that the three integrals are respectively equal to

$$\text{(i)} \quad \frac{(2n)!}{(2\pi)^{2n+1}} \left\{1 - \frac{1}{2^{2n+1}} + \frac{1}{4^{2n+1}} - \frac{1}{5^{2n+1}} + \&c.\right\},$$

$$\text{(ii)} \quad \frac{(2n)!}{\pi^{2n+1}} \left\{1 - \frac{1}{5^{2n+1}} + \frac{1}{7^{2n+1}} - \frac{1}{11^{2n+1}} + \&c.\right\},$$

$$\text{(iii)} \quad \frac{(2n)!}{(2\pi)^{2n+1}} \left\{1 + \frac{1}{2^{2n+1}} - \frac{1}{4^{2n+1}} - \frac{1}{5^{2n+1}} + \&c\right\}.$$

Now it is shown in the paper in the *Quarterly Journal* that

$$1 - \frac{1}{2^{2n+1}} + \frac{1}{4^{2n+1}} - \frac{1}{5^{2n+1}} + \&c. = \frac{1}{\sqrt{3}} \frac{I_n}{(2n)!} \left(\frac{2\pi}{3}\right)^{2n+1}, \text{ (§58, p. 35)},$$

$$1 - \frac{1}{5^{2n+1}} + \frac{1}{7^{2n+1}} - \frac{1}{11^{2n+1}} + \&c. = \frac{1}{\sqrt{3}} \frac{H_n}{(2n)!} \left(\frac{\pi}{3}\right)^{2n+1}, \text{ (§ 84, p. 49)},$$

$$1 + \frac{1}{2^{2n+1}} - \frac{1}{4^{2n+1}} - \frac{1}{5^{2n+1}} + \&c. = \frac{1}{\sqrt{3}} \frac{J_n}{(2n)!} \left(\frac{\pi}{3}\right)^{2n+1}, \text{ (§ 75, p. 44)}.$$

Substituting their values for these series, we obtain for the integrals (i), (ii), (iii) the values assigned to them in § 36.

Expressions for P_n, Q_n, R_n, T_n as definite integrals, §§ 38–39.

§ 38. The corresponding *P*- and *Q*-formulæ are

$$P_0 - \frac{P_1}{2!} a^2 + \frac{P_2}{4!} a^4 - \&c. = \frac{\cosh a}{\cosh 2a} = 2\sqrt{2} \int_0^\infty \frac{\cosh \pi t}{\cosh 2\pi t} \cos 4at \, dt,$$

$$Q_1 a - \frac{Q_2}{3!} a^3 + \frac{Q_3}{5!} a^5 - \&c. = \frac{\sinh a}{\cosh 2a} = 2\sqrt{2} \int_0^\infty \frac{\sinh \pi t}{\cosh 2\pi t} \sin 4at \, dt;$$

whence we find

$$\text{(i)} \qquad \int_0^\infty t^{2n} \frac{\cosh \pi t}{\cosh 2\pi t}\, dt = \frac{P_n \sqrt{2}}{4^{2n+1}},$$

$$\text{(ii)} \qquad \int_0^\infty t^{2n-1} \frac{\sinh \pi t}{\cosh 2\pi t}\, dt = \frac{Q_n \sqrt{2}}{4^{2n}}.$$

Similarly, from the formula

$$R_0 - \frac{R_1}{2!} a^2 + \frac{R_2}{4!} a^4 - \&c. = \frac{\cosh^2 a}{\cosh 3a}$$

$$= \tfrac{1}{2} \frac{e^a + e^{-a}}{e^{2a} + e^{-2a} - 1} = 3 \int_0^\infty \frac{\cosh 2\pi t}{\cosh 3\pi t} \cos 6at\, dt,$$

$$T_1 a - \frac{T_2}{3!} a^3 + \frac{T_3}{5!} a^5 - \&c. = \tfrac{1}{2} \frac{\sinh 2a}{\cosh 3a}$$

$$= \tfrac{1}{2} \frac{e^a - e^{-a}}{e^{3a} + e^{-2a} - 1} = \sqrt{3} \int_0^\infty \frac{\sinh 2\pi t}{\cosh 3\pi t} \sin 6at\, dt,$$

we find

$$\text{(iii)} \qquad \int_0^\infty t^{2n} \frac{\cosh 2\pi t}{\cosh 3\pi t}\, dt = \frac{2R_n}{6^{2n+1}},$$

$$\text{(iv)} \quad \int_0^\infty t^{2n-1} \frac{\sinh 2\pi t}{\cosh 3\pi t}\, dt = \int_0^\infty t^{2n-1} \frac{e^{\pi t} - e^{-\pi t}}{e^{2\pi t} + e^{-2\pi t} - 1}\, dt = \frac{2 T_n \sqrt{3}}{6^{2n}}.$$

§ 39. To obtain the values of the four integrals (i)–(iv) by expansion we notice that

$$\frac{\cosh x}{\cosh 3x} = e^{x} + e^{-3x} - e^{-5x} - e^{-7x} + \&c.,$$

$$\frac{\sinh x}{\cosh 3x} = e^{x} - e^{-3x} - e^{-5x} + e^{-7x} + \&c.,$$

$$\frac{\cosh 2x}{\cosh 3x} = e^{-x} + e^{-5x} - e^{-7x} - e^{-11x} + \&c.,$$

$$\frac{\sinh 2x}{\cosh 2x} = e^{-x} - e^{-5x} - e^{-7x} + e^{-11x} + \&c.,$$

so that the four integrals are respectively equal to

$$\text{(i)} \qquad \frac{(2n)!}{\pi^{2n+1}}\left\{1+\frac{1}{3^{2n+1}}-\frac{1}{5^{2n+1}}-\frac{1}{7^{2n+1}}+\&c.\right\},$$

$$\text{(ii)} \quad \frac{(2n-1)!}{\pi^{2n}}\left\{1-\frac{1}{3^{2n}}-\frac{1}{5^{2n}}+\frac{1}{7^{2n}}+\&c.\right\},$$

$$\text{(iii)} \quad \frac{(2n)!}{\pi^{2n+1}}\left\{1+\frac{1}{5^{2n}}-\frac{1}{7^{2n+1}}-\frac{1}{11^{2n+1}}+\&c.\right\},$$

$$\text{(iv)} \quad \frac{(2n-1)!}{\pi^{2n}}\left\{1-\frac{1}{5^{2n}}-\frac{1}{7^{2n}}+\frac{1}{11^{2n}}+\&c.\right\}.$$

Now, from the paper in the *Quarterly Journal*, we have

$$1+\frac{1}{3^{2n+1}}-\frac{1}{5^{2n+1}}-\frac{1}{7^{2n+1}}+\&c.=\frac{P_n\sqrt{2}}{(2n)!}\left(\frac{\pi}{4}\right)^{2n+1}, \ (\S\,109, \text{p. } 60),$$

$$1-\frac{1}{3^{2n}}-\frac{1}{5^{2n}}+\frac{1}{7^{2n}}+\&c.=\frac{Q_n\sqrt{2}}{(2n-1)!}\left(\frac{\pi}{4}\right)^{2n}, \ (\S\,119, \text{p. } 64),$$

$$1+\frac{1}{5^{2n+1}}-\frac{1}{7^{2n+1}}-\frac{1}{11^{2n+1}}+\&c.=\frac{2R_n}{(2n)!}\left(\frac{\pi}{6}\right)^{2n+1}, \ (\S\,132, \text{p. } 70),$$

$$1-\frac{1}{5^{2n}}-\frac{1}{7^{2n}}+\frac{1}{11^{2n}}+\&c.=\frac{2T_n\sqrt{3}}{(2n-1)!}\left(\frac{\pi}{6}\right)^{2n}, \ (\S\,143, \text{p. } 75);$$

and, substituting for these series their values, we obtain for the four integrals the values assigned to them in § 38.

The numbers I_n, H_n, J_n, P_n, &c., § 40.

§ 40. Although in the headings of §§ 33–38 the results have been described as expressing B_n, E_n, I_n, &c. by means of definite integrals, the equations may be more properly regarded as affording evaluations of the integrals by means of the known values of the numbers B_n, E_n, I_n, &c.

In the paper in the *Quarterly Journal* the quantities I_n, H_n, J_n, P_n, &c. which represent special values of the Bernoullian Function are treated as known systems of numbers of the same kind as B_n and E_n. Various formulæ are there given by means of which they may be calculated, as well as numerous other relations.

The forms of the integrals for the Bernoullian and Eulerian numbers, §§ 41–43.

§ 41. The integrals by which the Bernoullian and Eulerian numbers are represented exhibit a peculiarity of form which form which can scarcely have failed to be noticed before.

Thus, the Bernoullian numbers may be defined by the equation

$$\frac{1}{e^a - 1} = \frac{1}{a} - \tfrac{1}{2} + \Sigma_1^\infty (-1)^{n-1} \frac{B_n}{(2n)!} a^{2n-1},$$

and the expression for B_n as a definite integral is (§ 33)

$$\frac{B_n}{4n} = \int_0^\infty \frac{t^{2n-1}dt}{e^{2\pi t} - 1} = \frac{1}{(2\pi)^{2n}} \int_0^\infty \frac{t^{2n-1}dt}{e^t - 1}.$$

The peculiarity consists in the fact that a function of the same form, $\dfrac{1}{e^t - 1}$, as that by which the Bernoullian numbers are defined, occurs also under the integral sign.

§ 42. Similarly, in the case of the Eulerian numbers, we have

$$\frac{1}{\cosh a} = \Sigma_0^\infty (-1)^n \frac{E_n}{(2n)!} a^{2n},$$

and

$$E_n = 2^{2n+1} \int_0^\infty \frac{t^{2n}dt}{\cosh \pi t} = \frac{2^{2n+1}}{\pi^{2n+1}} \int_0^\infty \frac{t^{2n}dt}{\cosh t}.$$

Thus the form of the function, by which the Eulerian numbers are defined, persists also under the integral sign.

§ 43. A somewhat similar repetition of form occurs in the other definite integrals representing the Bernoullian numbers which may be derived from §§ 33 and 34.

Thus, from § 34, we may deduce the expansion

$$\frac{1}{e^a + 1} = \tfrac{1}{2} + \Sigma_1^\infty (-1)^n \frac{(2^{2n} - 1) B_n}{(2n)!} a^{2n-1},$$

and, from § 33, we have

$$(2^{2n} - 2) \frac{B_n}{4n} = 2^{2n} \int_0^\infty \frac{t^{2n-1}dt}{e^{2\pi t} + 1} = \frac{1}{\pi^{2n}} \int_0^\infty \frac{t^{2n-1}dt}{e^t + 1}.$$

Also, from § 33,

$$\frac{1}{\sinh a} = \frac{1}{a} + \Sigma_1^\infty (-1)^n \frac{(2^{2n}-2)B_n}{(2n)!} a^{2n-1},$$

and, from § 34, we have

$$\frac{(2^{2n}-1)B_n}{2n} = \int_0^\infty \frac{t^{2n-1}dt}{\sinh \pi t} = \frac{1}{\pi^{2n}} \int_0^\infty \frac{t^{2n-1}dt}{\sinh t}.$$

These formulæ exhibit a reciprocity of form, *i.e.*, when the coefficient is $(2^{2n}-1)B_n$, the corresponding integral represents $(2^{2n}-2)B_n$; and when the coefficient is $(2^{2n}-2)B_n$, the corresponding integral represents $(2^{2n}-1)B_n$.

The forms of the integrals for the I-, H-, J-numbers, §§ 44–45

§ 44. The *I*-formulæ show a persistence of form under the integral sign similar to that which occurs in the integral which represents the Eulerian numbers. This is not the case with the formulæ relating to the *H*- and *J*-numbers; which exhibit, however, a reciprocity of the same kind as that which has just been noticed in § 43.

Thus, taking the *I*-formulæ, we have

$$\frac{\sinh \frac{1}{2}a}{\sinh \frac{3}{2}a} = \frac{1}{e^a + e^{-a} + 1} = \tfrac{2}{3}\Sigma_0^\infty (-1)^n \frac{I_n}{(2n)!} a^{2n},$$

and

$$I_n = 3^{2n+1}\sqrt{3}\int_0^\infty t^{2n}\frac{\sinh \pi t}{\sinh 3\pi t}dt = \sqrt{3}\int_0^\infty t^{2n}\frac{\sinh \frac{1}{3}\pi t}{\sinh \pi t}dt$$

$$= \frac{3^{2n+1}\sqrt{3}}{(2\pi)^{2n+1}}\int_0^\infty t^{2n}\frac{\sinh \frac{1}{2}t}{\sinh \frac{3}{2}t}dt = \frac{3^{2n+1}\sqrt{3}}{(2\pi)^{2n+1}}\int_0^\infty \frac{t^{2n}dt}{e^t + e^{-t} + 1}.$$

§ 45. In the case of the *H*- and *J*-formulæ, we have

$$\frac{\cosh \frac{1}{2}a}{\cosh \frac{3}{2}a} = \frac{1}{e^a + e^{-a} - 1} = \tfrac{2}{3}\Sigma_0^\infty (-1)^n \frac{H_n}{(2n)!} a^{2n},$$

$$\frac{\sinh 2a}{\sinh 3a} = \frac{e^a + e^{-a}}{e^{2a} + e^{-2a} + 1} = \tfrac{1}{3}\Sigma_0^\infty (-1)^n \frac{J_n}{(2n)!} a^{2n},$$

and

$$H_n = 3^{2n+1}\sqrt{3}\int_0^\infty t^{2n}\frac{\sinh 2\pi t}{\sinh 3\pi t}\,dt = \sqrt{3}\int_0^\infty t^{2n}\frac{\sinh \frac{2}{3}\pi t}{\sinh \pi t}\,dt$$

$$= \frac{3^{2n+1}\sqrt{3}}{\pi^{2n+1}}\int_0^\infty t^{2n}\frac{\sinh 2t}{\sinh 3t}\,dt = \frac{3^{2n+1}\sqrt{3}}{\pi^{2n+1}}\int_0^\infty t^{2n}\frac{e^t+e^{-t}}{e^{2t}+e^{-2t}+1}\,dt,$$

$$J_n = 6^{2n+1}\sqrt{3}\int_0^\infty t^{2n}\frac{\cosh \pi t}{\cosh 3\pi t}\,dt = \sqrt{3}\int_0^\infty t^{2n}\frac{\cosh \frac{1}{6}\pi t}{\cosh \frac{1}{2}\pi t}\,dt$$

$$= \frac{3^{2n+1}\sqrt{3}}{\pi^{2n+1}}\int_0^\infty t^{2n}\frac{\cosh \frac{1}{2}t}{\cosh \frac{3}{2}t}\,dt = \frac{3^{2n+1}\sqrt{3}}{\pi^{2n+1}}\int_0^\infty \frac{t^{2n}\,dt}{e^t+e^{-t}-1}.$$

The forms of the integrals for the P-, Q-, R-, T-numbers, §§ 46–50.

§ 46. The form persists under the integral sign in the case of the P- and Q- formulæ: for we have

$$\frac{\cosh a}{\cosh 2a} = \Sigma_0^\infty (-1)^n \frac{P_n}{(2n)!}a^{2n},$$

$$P_n = \frac{4^{2n+1}}{\sqrt{2}}\int_0^\infty t^{2n}\frac{\cosh 2\pi t}{\cosh \pi t}\,dt = \frac{4^{2n+1}}{\pi^{2n+1}\sqrt{2}}\int_0^\infty t^{2n}\frac{\cosh t}{\cosh 2t}\,dt,$$

and

$$\frac{\sinh a}{\cosh 2a} = \Sigma_1^\infty (-1)^{n-1}\frac{Q_n}{(2n-1)!}a^{2n-1},$$

$$Q_n = \frac{4^{2n}}{\sqrt{2}}\int_0^\infty t^{2n-1}\frac{\sinh \pi t}{\cosh 2\pi t}\,dt = \frac{4^{2n}}{\pi^{2n}\sqrt{2}}\int_0^\infty t^{2n-1}\frac{\sinh t}{\cosh 2t}\,dt.$$

§ 47. The persistence also occurs in the case of the T-numbers, for

$$\frac{1}{2}\frac{\sinh 2a}{\cosh 3a} = \Sigma_1^\infty (-1)^{n-1}\frac{T_n}{(2n-1)!}a^{2n-1},$$

$$T_n = \frac{6^{2n}}{2\sqrt{3}}\int_0^\infty t^{2n-1}\frac{\sinh 2\pi t}{\cosh 3\pi t}\,dt = \frac{6^{2n}}{2\pi^{2n}\sqrt{3}}\int_0^\infty t^{2n-1}\frac{\sinh 2t}{\sinh 3t}\,dt.$$

§ 48. In the case of the R-numbers, the form of the function under the integral sign is different; for we have

$$\frac{\cosh^2 a}{\cosh 3a} = \Sigma_0^\infty (-1)^n \frac{R_n}{(2n)!} a^{2n},$$

and

$$R_n = \frac{6^{2n+1}}{2} \int_0^\infty t^{2n} \frac{\cosh 2\pi t}{\cosh 3\pi t} dt = \frac{6^{2n+1}}{2\pi^{2n+1}} \int_0^\infty t^{2n} \frac{\cosh 2t}{\cosh 3t} dt.$$

§ 49. As mentioned in § 32 the R-numbers do not form an independent system, being connected with the Eulerian numbers by the relation

$$R_n = \frac{3^{2n+1}+1}{4} E_n$$

Thus we may write the expansion-equation in the form

$$\frac{\cosh 2a}{\cosh 3a} + \frac{1}{\cosh 3a} = \tfrac{1}{2} \Sigma_0^\infty (-1)^n (3^{2n+1}+1) \frac{E_n}{(2n)!} a^{2n},$$

whence

$$\frac{\cosh 2a}{\cosh 3a} = \tfrac{1}{2} \Sigma_0^\infty (-1)^n (3^{2n}+1) \frac{E_n}{(2n)!} a^{2n}.$$

§ 50. The pair of R-formulæ may therefore be written

$$\frac{\cosh 2a}{\cosh 3a} = \tfrac{1}{2} \Sigma_0^\infty (-1)^n \frac{(3^{2n}+1) E_n}{(2n)!} a^{2n},$$

$$(3^{2n+1}+1) E_n = 2.6^{2n+1} \int_0^\infty t^{2n} \frac{\cosh 2\pi t}{\cosh 3\pi t} dt = \frac{2.6^{2n+1}}{\pi^{2n+1}} \int_0^\infty t^{2n} \frac{\cosh 2t}{\cosh 3t} dt.$$

It may be noticed that we also have the pair of formulæ:

$$\frac{\cosh^2 a}{\cosh 3a} = \tfrac{1}{4} \Sigma_0^\infty (-1)^n \frac{(3^{2n+1}+1) E_n}{(2n)!} a^{2n},$$

$$(3^{2n}+1) E_n = \tfrac{4}{3}.6^{2n+1} \int_0^\infty t^{2n} \frac{\cosh^2 \pi t}{\cosh 3\pi t} dt = \frac{4.6^{2n+1}}{3\pi^{2n+1}} \int_0^\infty t^{2n} \frac{\cosh^2 t}{\cosh 3t} dt.$$

In both cases that is a reciprocity between $(3^{2n}+1) E_n$ and $(3^{2n+1}+1) E_n$.

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The reciprocity of form, §§ 51–54.

§ 51. The manner in which the integrals representing the Bernoullian and other numbers have been obtained in this paper affords a general explanation of the reciprocity and similarity of form which have been noticed in the preceding sections. For these integrals were derived from the more general integrals involving $\cos at$ and $\sin at$ in § 26, and these latter formulæ necessarily exhibit a reciprocity by virtue of the reciprocal equations of § 15. Thus we see that there must always be a reciprocity in the expressions for the coefficients, or numbers, as integrals, and that in special cases this reciprocity may ~~become a persistence~~.

§ 52. In general, if

$$f(a)=\Sigma_0^\infty (-1)^n \frac{g_n}{(2n)!} a^{2n}, \quad \phi(a)=\Sigma_0^\infty (-1)^n \frac{h_n}{(2n)!} a^{2n},$$

and if the ~~functions~~ f and ϕ are connected by the relation

$$f(a)=\int_0^\infty \phi(t)\cos at\,dt,$$

then
$$g_n=\int_0^\infty t^{2n}\phi(t)\,dt, \quad h_n=\frac{2}{\pi}\int_0^\infty t^{2n} f(t)\,dt.$$

For example, taking the formula,

$$\int_0^\infty \frac{\sinh 2\pi t}{\sinh 3\pi t}\cos at\,dt=\frac{1}{2\sqrt{3}}\,\frac{\cosh \frac{1}{6}a}{\cosh \frac{1}{2}a},$$

we have

$$f(a)=\frac{1}{2\sqrt{3}}\,\frac{\cosh \frac{1}{6}a}{\cosh \frac{1}{2}a}=\frac{1}{2\sqrt{3}}\,\tfrac{2}{3}\Sigma_0^\infty (-1)^n \frac{H_n}{(2n)!}\left(\frac{a}{3}\right)^{2n},$$

$$\phi(a)=\frac{\sinh 2\pi a}{\sinh 3\pi a}=\tfrac{1}{3}\Sigma_0^\infty (-1)^n \frac{J_n}{(2n)!}(\pi a)^{2n}.$$

and the formulæ giving the coefficients as integrals are

$$\frac{1}{3\sqrt{3}}\,\frac{H_n}{3^{2n}}=\int_0^\infty t^{2n}\frac{\sinh 2\pi t}{\sinh 3\pi t}\,dt,$$

$$\tfrac{1}{3}J_n\pi^{2n}=\frac{2}{\pi}\int_0^\infty t^{2n}\,\frac{1}{2\sqrt{3}}\,\frac{\cosh \frac{1}{6}t}{\cosh \frac{1}{2}t}\,dt,$$

which are readily identified with the values of H_n and J_n given in § 45.

§ 53. Similarly, if

$$f(a)=\Sigma_1^\infty(-1)^{n-1}\frac{j_n}{(2n-1)!}a^{2n-1},\quad \phi(a)=\Sigma_1^\infty(-1)^{n-1}\frac{k_n}{(2n-1)!}a^{2n-1},$$

and if f and ϕ are connected by the relation

$$f(a)=\int_0^\infty \phi(t)\sin at\,dt,$$

then

$$j_n=\int_0^\infty t^{2n-1}\phi(t)\,dt,\quad k_n=\frac{2}{\pi}\int_0^\infty t^{2n-1}f(t)\,dt.$$

§ 54. Thus, when we derive the expressions for the numbers B_n, E_n, I_n, &c. as definite integrals from the integrals involving $\cos at$ and $\sin at$ by equating the coefficients of t^{2n} and t^{2n-1}, we see that the reciprocity of form is a necessary consequence of the reciprocity of f and ϕ in the formulæ of § 15; but when we evaluate the integrals by expanding the quantity under the integral sign in powers of $e^{-\pi t}$ and integrating each term separately (which seems the most natural method of procedure), there is no apparent reason for any similarity of form between the generating functions of the different classes of numbers and their expressions as definite integrals.

The persistence of form takes place when the functions f and ϕ happen to be the same.

Another method of obtaining the integrals for B_n, E_n, I_n, &c., §§ 55–57.

§ 55. The results obtained in §§ 33–38 might have been derived directly from the general expressions for $A_n'(x)$ and $A_n(x)$ as definite integrals (§§ 4 and 9) by means of the special values of the Bernoullian function which were referred to in § 40.

These special values of $A_n'(x)$ are:

$$A'_{2n}(\tfrac{1}{2})=0,$$

$$2^{2n+1}A'_{2n+1}(\tfrac{1}{2})=(-1)^n E_n,$$

$$3^{2n}A'_{2n}(\tfrac{1}{3})=(-1)^n(2^{2n}-1)(3^{2n}-3)\frac{B_n}{4n},$$

$$3^{2n+1}A'_{2n+1}(\tfrac{1}{3})=(-1)^n H_n,$$

$$4^{2n}A'_{2n}(\tfrac{1}{4})=(-1)^n 2Q_n,$$

$$4^{2n+1}A'_{2n+1}(\tfrac{1}{4})=(-1)^n 2P_n,$$

$$6^{2n}A'_{2n}(\tfrac{1}{6})=(-1)^n 6T_n,$$

$$6^{2n+1}A'_{2n+1}(\tfrac{1}{6})=(-1)^n\frac{3^{2n+1}+3}{2}E_n,$$

and the special values of $A_n(x)$ are

$$2^{2n}A_{2n}(\tfrac{1}{2}) = (-1)^n(2^{2n}-2)\frac{B_n}{2n},$$

$$A_{2n+1}(\tfrac{1}{2}) = 0,$$

$$3^{2n}A_{2n}(\tfrac{1}{3}) = (-1)^n(3^{2n}-3)\frac{B_n}{2n},$$

$$3^{2n+1}A_{2n+1}(\tfrac{1}{3}) = (-1)^{n+1}I_n,$$

$$4^{2n}A_{2n}(\tfrac{1}{4}) = (-1)^n(2^{2n}-2)\frac{B_n}{2n},$$

$$4^{2n+1}A_{2n+1}(\tfrac{1}{4}) = (-1)^{n+1}E_n,$$

$$6^{2n}A_{2n}(\tfrac{1}{6}) = (-1)^{n-1}(2^{2n}-2)(3^{2n}-3)\frac{B_n}{4n},$$

$$6^{2n+1}A_{2n+1}(\tfrac{1}{6}) = (-1)^{n+1}J_n.$$

These systems of values are taken from § 205 (p. 107) and § 252 (p. 131) of the paper in the *Quarterly Journal.*

§ 56. Besides the formulæ in §§ 33–38 these equations also give the results

$$(3^{2n}-3)\frac{B_n}{4n} = 3^{2n}\int_0^\infty t^{2n-1}\left(1-\frac{2\sinh 2\pi t}{2\cosh 2\pi t+1}\right)dt,$$

$$(2^{2n}-2)(3^{2n}-3)\frac{B_n}{4n} = -6^{2n}\int_0^\infty t^{2n-1}\left(1-\frac{\sinh 2\pi t}{2\cosh 2\pi t-1}\right)dt.$$

§ 57. These formulæ may be easily verified by expansion; for the former integral

$$= \int_0^\infty t^{2n-1}\left(1-\coth 3\pi t+\frac{\cosh \pi t}{\sinh 3\pi t}\right)dt.$$

Now $\quad 1-\coth 3\pi t = -2e^{-6\pi t}-2e^{-12\pi t}-2e^{-18\pi t}-\&c.,$

and $\quad \dfrac{\cosh \pi t}{\sinh 3\pi t} = e^{-2\pi t}+e^{-4\pi t}+e^{-8\pi t}+e^{-10\pi t}+\&c.$

Thus the integral

$$= \frac{(2n-1)!}{\pi^{2n}}\left\{\frac{1}{2^{2n}}+\frac{1}{4^{2n}}+\frac{1}{6^{2n}}+\frac{1}{8^{2n}}+\&c.\right.$$
$$\left.-3\left(\frac{1}{6^{2n}}+\frac{1}{12^{2n}}+\frac{1}{18^{2n}}+\frac{1}{24^{2n}}+\&c.\right)\right\}$$

$$= \frac{(2n-1)!}{\pi^{2n}}\,\tfrac{1}{2}\,\frac{B_n\pi^{2n}}{(2n)!}\left(1-\frac{3}{3^{2n}}\right) = \frac{3^{2n}-3}{3^{2n}}\,\frac{B_n}{4n}.$$

Similarly, the second integral

$$= \int_0^\infty t^{2n-1}\left(1 - \tanh 3\pi t - \frac{\sinh \pi t}{\cosh 3\pi t}\right) dt$$

$$= \frac{(2n-1)}{\pi^{2n}}\left\{2\left(\frac{1}{6^{2n}} - \frac{1}{12^{2n}} + \frac{1}{18^{2n}} - \frac{1}{24^{2n}} + \&c.\right)\right.$$

$$\left. - \left(\frac{1}{2^{2n}} - \frac{1}{4^{2n}} - \frac{1}{8^{2n}} + \frac{1}{10^{2n}} + \&c\right)\right\}$$

$$= \frac{(2^{2n-1}-1)\,B_n}{2n}\left\{\frac{2}{6^{2n}} - \frac{1}{2^{2n}}\left(1 - \frac{1}{3^{2n}}\right)\right\} = -\frac{(2^{2n}-2)(3^{2n}-3)}{6^{2n}}\,\frac{B_n}{4n}.$$

Other expressions for $A_n'(x)$ and $A_n(x)$ as definite integrals, §§ 58–64.

§ 58. We may also express as definite integrals the sums of the sine- and cosine-series in §§ 2 and 8 by a slightly different process from that employed in § 3.

For we have

$$e^{-t}\cos \pi x + \tfrac{1}{3}e^{-3t}\cos 3\pi x + \tfrac{1}{5}e^{-5t}\cos 5\pi x + \&c.$$

$$= \tfrac{1}{4}\log\frac{1 + 2e^{-t}\cos \pi x + e^{-2t}}{1 - 2e^{-t}\cos \pi x + e^{-2t}};$$

whence, multiplying by t^{2n-2} and integrating,

$$(2n-2)!\left\{\cos \pi x + \frac{\cos 3\pi x}{3^{2n}} + \frac{\cos 5\pi x}{5^{2n}} + \&c.\right\}$$

$$= \tfrac{1}{4}\int_0^\infty t^{2n-2}\log\frac{1 + 2e^{-t}\cos \pi x + e^{-2t}}{1 - 2e^{-t}\cos \pi x + e^{-2t}}\,dt.$$

Therefore, from (ii) of § 2,

$$\text{(i)}\quad \int_0^\infty t^{2n-2}\log\frac{1 + 2e^{-t}\cos \pi x + e^{-2t}}{1 - 2e^{-t}\cos \pi x + e^{-2t}}\,dt = (-1)^n\frac{2\pi^{2n}}{2n-1}A_{2n}'(x).$$

This result may be written also in the form

$$\int_0^\infty t^{2n-2}\log\frac{\cosh t + \cos \pi x}{\cosh t - \cos \pi x}\,dt = (-1)^n\frac{2\pi^{2n}}{2n-1}A_{2n}'(x),$$

or

$$\int_0^\infty t^{2n-2}\log\frac{\cosh \pi t + \cos \pi x}{\cosh \pi t - \cos \pi x}\,dt = (-1)^n\frac{2\pi}{2n-1}A_{2n}'(x).$$

§ 59. Similarly, by means of the summation

$$e^{-t}\sin\pi x+\tfrac{1}{3}e^{-3t}\sin 3\pi x+\tfrac{1}{5}e^{-5t}\sin 5\pi x+\&c.$$
$$=\tfrac{1}{2}\tan^{-1}\frac{2e^{-t}\sin\pi x}{1-e^{-2t}\cos 2\pi x},$$

we find

$$(2n-1)!\left\{\sin\pi x+\frac{\sin 3\pi x}{3^{2n+1}}+\frac{\sin 5\pi x}{5^{2n+1}}+\&c.\right\}$$
$$=\tfrac{1}{2}\int_0^\infty t^{2n-1}\tan^{-1}\frac{2e^{-t}\sin\pi x}{1-e^{-2t}\cos 2\pi x}\,dt;$$

whence, from (1) of § 2,

$$\text{(ii)}\quad \int_0^\infty t^{2n-1}\tan^{-1}\frac{2e^{-t}\sin 2\pi x}{1-e^{-2t}\cos 2\pi x}\,dt=(-1)^n\frac{\pi^{2n+1}}{2n}A'_{2n+1}(x).$$

§ 60. Applying the summations

$$e^{-t}\cos 2\pi x+\tfrac{1}{2}e^{-2t}\cos 4\pi x+\tfrac{1}{3}e^{-3t}\cos 6\pi x+\&c.$$
$$=-\tfrac{1}{2}\log(1-2e^{-t}\cos 2\pi x+e^{-2t}),$$
$$e^{-t}\sin 2\pi x+\tfrac{1}{2}e^{-2t}\sin 4\pi x+\tfrac{1}{3}e^{-3t}\sin 6\pi x+\&c.$$
$$=\tan^{-1}\frac{e^{-t}\sin 2\pi x}{1-e^{-t}\cos 2\pi x},$$

in the same manner to the formulæ (ii) and (i) of § 8, we find

$$\text{(iii)}\quad \int_0^\infty t^{2n-2}\log(1-2^{-t}\cos 2e^{-t}\cos 2\pi x+e^{-2t})\,dt=(-1)^n\frac{2^{2n}\pi^{2n}}{2n-1}A_{2n}(x),$$

$$\text{(iv)}\quad \int_0^\infty t^{2n-1}\tan^{-1}\frac{e^{-t}\sin 2\pi x}{1-e^{-t}\cos 2\pi x}\,dt=(-1)^{n+1}\frac{2^{2n}\pi^{2n+1}}{2n}A_{2n+1}(x).$$

§ 61. As particular cases, putting $x=\frac{1}{2},\ \frac{1}{3},\ \frac{1}{6}$ in (iii), we have

$$\int_0^\infty t^{2n-2}\log(1+e^{-t})\,dt=\frac{(2^{2n-1}-1)B_n}{2n(2n-1)}\pi^{2n},$$

$$\int_0^\infty t^{2n-2}\log(1+e^{-t}+e^{-2t})\,dt=\tfrac{1}{2}\frac{(3^{2n}-3)B_n}{2n(2n-1)}\left(\frac{2\pi}{3}\right)^{2n},$$

$$\int_0^\infty t^{2n-2}\log(1-e^{-t}+e^{-2t})\,dt=-\tfrac{1}{2}\frac{(2^{2n}-2)(3^{2n}-3)B_n}{2n(2n-1)}\left(\frac{\pi}{3}\right)^{2n}.$$

By putting $x=\frac{1}{4}$, we merely reproduce the first result.

§ 62. Similarly, putting $x=\frac{1}{4}, \frac{1}{3}, \frac{1}{6}$ in (iv), we have

$$\int_0^\infty t^{2n-1}\tan^{-1}(e^{-t})\,dt=\frac{E_n}{4n}\left(\frac{\pi}{2}\right)^{2n+1},$$

$$\int_0^\infty t^{2n-1}\tan^{-1}\left(\frac{\sqrt{3}}{2e^t+1}\right)dt=\frac{I_n}{4n}\left(\frac{2\pi}{3}\right)^{2n+1},$$

$$\int_0^\infty t^{2n-1}\tan^{-1}\left(\frac{\sqrt{3}}{2e^t-1}\right)dt=\frac{J_n}{4n}\left(\frac{\pi}{3}\right)^{2n+1}.$$

§ 63. We may also obtain other expressions for $A_n'(x)$ and $A_n(x)$ as definite integrals by starting with different expansions in sines and cosines.

Thus, taking the expansion

$$e^{-t}\cos\pi x+3e^{-3t}\cos 3\pi x+5e^{-5t}\cos 5\pi x+\&c.$$

$$=2\cosh t\cos\pi x\,\frac{\sinh^2 t-\sin^2\pi x}{(\cosh 2t-\cos 2\pi x)^2},$$

and multiplying by t^{2n} and integrating, we find

$$(2n)!\left\{\cos\pi x+\frac{\cos 3\pi x}{3^{2n}}+\frac{\cos 5\pi x}{5^{2n}}+\&c.\right\}$$

$$=2\int_0^\infty t^{2n}\,\frac{\cosh t\cos\pi x\,(\sinh^2 t-\sin^2\pi x)}{(\cosh 2t-\cos 2\pi x)^2}\,dt;$$

whence

$$4\pi\int_0^\infty t^{2n}\,\frac{\cosh\pi t\cos\pi x\,(\sinh^2\pi t-\sin^2\pi x)}{(\cosh 2\pi t-\cos 2\pi x)^2}\,dt=(-1)^n\,2nA_{2n}'(x).$$

§ 64. All these results may be deduced by integration by parts from the original expressions for $A_n'(x)$ and $A_n(x)$ as definite integrals which were given in §§ 4 and 9.

A continuation of the present paper, which will appear in the next volume of the *Messenger*, contains a discussion of other integrals connected with the Bernoullian Function, including integrals of the forms $\int A_m'(x)\,A_n'(x)\,dx$, &c., $\int A_n'(x)\cos mx\,dx$, &c., $\int\frac{A_n'(x)}{\sin\pi x}\,dx$, &c.

ON CERTAIN EXACT SYSTEMS OF PFAFFIAN EQUATIONS OF A SPECIAL TYPE.

By *J. Brill, M.A.*, St. John's College.

1. I PROPOSE to discuss a set of Pfaffian equations containing $2n-1$ variables, there being n dependent variables $u_1, u_2, \ldots, u_n$, and $n-1$ independent variables $\phi_1, \phi_2, \ldots, \phi_{n-1}$. Thus there will be n equations, and I shall assume these equations to be of the type

$$du_i = \sum_{j=1}^{n-1} \{(\sum_{k=1}^{n} \gamma_{ijk} u_k)\, d\phi_j\},$$

where the γ's are constants, the n equations being obtained by assigning to i all integral values from 1 to n inclusive.

A complete discussion of this system of equations would be eminently desirable, but, in the present state of our knowledge, such a discussion is only practicable in the case in which the equations form an exact system.

By eliminating the $d\phi$'s we obtain a single Pfaffian equation

$$U_1 du_1 + U_2 du_2 + \ldots + U_n du_n = 0,$$

containing the u's only, the U's being homogeneous integral functions of the u's of degree $n-1$. Thus, in the general case, the u's will be connected by one or more relations not involving the ϕ's. Subsequently it will be found that in the case in which our system of equations is exact there will be a single relation involving the u's only. In that case the above equation will be capable of being made exact by the introduction of a factor.

2. The case of three variables, two dependent and one independent, is merely a question of ordinary differential equations, and need not detain us here. So we pass on at once to the case of three dependent and two independent variables. In this case we shall write our equations in the form

$$\left.\begin{aligned} du &= (a_1 u + b_1 v + c_1 w)\, d\phi + (\alpha_1 u + \beta_1 v + \gamma_1 w)\, d\psi \\ dv &= (a_2 u + b_2 v + c_2 w)\, d\phi + (\alpha_2 u + \beta_2 v + \gamma_2 w)\, d\psi \\ dw &= (a_3 u + b_3 v + c_3 w)\, d\phi + (\alpha_3 u + \beta_3 v + \gamma_3 w)\, d\psi \end{aligned}\right\} \ldots(1).$$

If we proceed to apply the Jacobian test of integrability to these equations, we shall obtain for the two fundamental operators

$$\Delta_1 = (a_1u + b_1v + c_1w)\frac{\partial}{\partial u} + (a_2u + b_2v + c_2w)\frac{\partial}{\partial v} + (a_3u + b_3v + c_3w)\frac{\partial}{\partial w} + \frac{\partial}{\partial \phi}$$

and

$$\Delta_2 = (\alpha_1u + \beta_1v + \gamma_1w)\frac{\partial}{\partial u} + (\alpha_2u + \beta_2v + \gamma_2w)\frac{\partial}{\partial v} + (\alpha_3u + \beta_3v + \gamma_3w)\frac{\partial}{\partial w} + \frac{\partial}{\partial \psi}.$$

Thus, writing

$$\begin{aligned}\Delta_3 &= \Delta_2\Delta_1 - \Delta_1\Delta_2 \\ &= (A_1u + B_1v + C_1w)\frac{\partial}{\partial u} + (A_2u + B_2v + C_2w)\frac{\partial}{\partial v} \\ &\quad + (A_3u + B_3v + C_3w)\frac{\partial}{\partial w},\end{aligned}$$

the law of formation of the A's, B's, and C's will be readily obtained; and it will be found that, if we denote the matrices

$$\begin{matrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{matrix} \quad \text{and} \quad \begin{matrix} \alpha_1, & \beta_1, & \gamma_1 \\ \alpha_2, & \beta_2, & \gamma_2 \\ \alpha_3, & \beta_3, & \gamma_3 \end{matrix}$$

by the symbols m and m' respectively, then the matrix

$$\begin{matrix} A_1, & B_1, & C_1 \\ A_2, & B_2, & C_2 \\ A_3, & B_3, & C_3 \end{matrix}$$

will be equivalent to $mm' - m'm$.

In order that Δ_3 may vanish identically, the A's, B's, and C's must be severally zero. In that case we should have

$$mm' - m'm = 0,$$

or the matrices m and m' would be convertible. This is, therefore, the condition that our system of equations should be exact; in which case its solution would consist of three exact integrals, from which u, v, w could be determined as functions of ϕ and ψ.

3. Introducing two arbitrary multipliers p and q, we deduce from equations (1) the single Pfaffian equation

$$du + pdv + qdw - \{(a_1 + pa_2 + qa_3)\, u + (b_1 + pb_2 + qb_3)\, v$$
$$+ (c_1 + pc_2 + qc_3)\, w\}\, d\phi - \{(\alpha_1 + p\alpha_2 + q\alpha_3)\, u$$
$$+ (\beta_1 + p\beta_2 + q\beta_3)\, v + (\gamma_1 + p\gamma_2 + q\gamma_3)\, w\}\, d\psi = 0 \quad \ldots.(2).$$

We proceed to calculate the system of Pfaffians and allied functions connected with this equation, as far as is necessary for our present purpose. We have

$$[01] = 1, \quad [02] = p, \quad [03] = q,$$

$$[04] = -\{(a_1 + pa_2 + qa_3)\, u + (b_1 + pb_2 + qb_3)\, v + (c_1 + pc_2 + qc_3)\, w\},$$

$$[05] = -\{(\alpha_1 + p\alpha_2 + q\alpha_3)\, u + (\beta_1 + p\beta_2 + q\beta_3)\, v + (\gamma_1 + p\gamma_2 + q\gamma_3)\, w\}.$$

Thus the first set of Pfaffians will be

$$[12] = 0, \quad [13] = 0, \quad [23] = 0, \quad [45] = 0,$$
$$[14] = a_1 + pa_2 + qa_3, \quad [15] = \alpha_1 + p\alpha_2 + q\alpha_3,$$
$$[24] = b_1 + pb_2 + qb_3, \quad [25] = \beta_1 + p\beta_2 + q\beta_3,$$
$$[34] = c_1 + pc_2 + qc_3, \quad [35] = \gamma_1 + p\gamma_2 + q\gamma_3.$$

Hence the first set of allied functions will be

$$[0123] = 0,$$

$$[0124] = b_1 + pb_2 + qb_3 - p\,(a_1 + pa_2 + qa_3),$$

$$[0125] = \beta_1 + p\beta_2 + q\beta_3 - p\,(\alpha_1 + p\alpha_2 + q\alpha_3),$$

$$[0134] = c_1 + pc_2 + qc_3 - q\,(a_1 + pa_2 + qa_3),$$

$$[0135] = \gamma_1 + p\gamma_2 + q\gamma_3 - q\,(\alpha_1 + p\alpha_2 + q\alpha_3),$$

$$[0234] = p\,(c_1 + pc_2 + qc_3) - q\,(b_1 + pb_2 + qb_3),$$

$$[0235] = p\,(\gamma_1 + p\gamma_2 + q\gamma_3) - q\,(\beta_1 + p\beta_2 + q\beta_3),$$

$$[0145] = \{(a_1 + pa_2 + qa_3)\, u + (b_1 + pb_2 + qb_3)\, v + (c_1 + pc_2 + qc_3)\, w\}\, (\alpha_1 + p\alpha_2 + q\alpha_3) - \{(\alpha_1 + p\alpha_2 + q\alpha_3)\, u + (\beta_1 + p\beta_2 + q\beta_3)\, v + (\gamma_1 + p\gamma_2 + q\gamma_3)\, w\}\, (a_1 + pa_2 + qa_3),$$

$$[0245] = \{(a_1 + pa_2 + qa_3)\, u + (b_1 + pb_2 + qb_3)\, v + (c_1 + pc_2 + qc_3)\, w\}\, (\beta_1 + p\beta_2 + q\beta_3) - \{(\alpha_1 + p\alpha_2 + q\alpha_3)\, u + (\beta_1 + p\beta_2 + q\beta_3)\, v + (\gamma_1 + p\gamma_2 + q\gamma_3)\, w\}\, (b_1 + pb_2 + qb_3),$$

$$[0345] = \{(a_1 + pa_2 + qa_3)\, u + (b_1 + pb_2 + qb_3)\, v + (c_1 + pc_2 + qc_3)\, w\}\, (\gamma_1 + p\gamma_2 + q\gamma_3) - \{(\alpha_1 + p\alpha_2 + q\alpha_3)\, u + (\beta_1 + p\beta_2 + q\beta_3)\, v + (\gamma_1 + p\gamma_2 + q\gamma_3)\, w\}\, (c_1 + pc_2 + qc_3).$$

Now, in order that (2) may be capable of being rendered exact by the introduction of a factor, it is necessary that the whole of this latter set of functions should vanish. But [0123] already vanishes, and the vanishing of [0124], [0125], [0134], [0135] requires the existence of the conditions

$$a_1 + pa_2 + qa_3 = \frac{b_1 + pb_2 + qb_3}{p} = \frac{c_1 + pc_2 + qc_3}{q} \quad \ldots (3),$$

$$\alpha_1 + p\alpha_2 + q\alpha_3 = \frac{\beta_1 + p\beta_2 + q\beta_3}{p} = \frac{\gamma_1 + p\gamma_2 + q\gamma_3}{q} \quad \ldots (4).$$

It is clear that these conditions also secure the vanishing of [0234], [0235], [0145], [0245], [0345].

The conditions (3) and (4) amount to four equations connecting two arbitrary quantities, and thus imply relations among the constants contained in equations (1). It will be subsequently seen that these conditions are consistent with those that express that our original system of equations should be exact.

4. We have shewn that when our system of equations is exact the matrices m and m' will be convertible, and will therefore be connected by a relation of the form

$$m' = Am^2 + Bm + C \quad \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

where A, B, C are ordinary algebraical quantities. Further, if we eliminate m from its characteristic equation with the

aid of (5), the resulting equation will be the characteristic equation of m'. Since the matrices are convertible, this process will not essentially differ from that of eliminating λ from the equation giving the latent roots of m, by means of the equation

$$\mu = A\lambda^2 + B\lambda + C,$$

where λ stands for a latent root of m. Thus, in this latter case, the resulting equation in μ will be identical with that giving the latent roots of m'. The latent roots of the two matrices are therefore connected in pairs by a relation exactly similar to that connecting the matrices themselves. In the sequel we shall distinguish corresponding latent roots with the aid of the same subscripts.

Now, by means of (5), we may express the elements of the matrix m' in terms of those of m. We thus obtain

$$\left.\begin{aligned} \alpha_1 &= A\,(a_1^2 + b_1a_2 + c_1a_3) + Ba_1 + C \\ \beta_1 &= A\,(a_1b_1 + b_1b_2 + c_1b_3) + Bb_1 \\ \gamma_1 &= A\,(a_1c_1 + b_1c_2 + c_1c_3) + Bc_1 \end{aligned}\right\} \quad\ldots\ldots(6),$$

$$\left.\begin{aligned} \alpha_2 &= A\,(a_2a_1 + b_2a_2 + c_2a_3) + Ba_2 \\ \beta_2 &= A\,(a_2b_1 + b_2^2 + c_2b_3) + Bb_2 + C \\ \gamma_2 &= A\,(a_2c_1 + b_2c_2 + c_2c_3) + Bc_2 \end{aligned}\right\} \quad\ldots\ldots(7),$$

$$\left.\begin{aligned} \alpha_3 &= A\,(a_3a_1 + b_3a_2 + c_3a_3) + Ba_3 \\ \beta_3 &= A\,(a_3b_1 + b_3b_2 + c_3b_3) + Bb_3 \\ \gamma_3 &= A\,(a_3c_1 + b_3c_2 + c_3^2) + Bc_3 + C \end{aligned}\right\} \quad\ldots\ldots(8).$$

It is thus seen that in the case in which equations (1) form an exact system, they virtually involve twelve constants. In this case we shall shew that equation (2) is capable of being rendered exact by the introduction of a factor.

Writing

$$a_1 + pa_2 + qa_3 = \lambda,$$

$$\alpha_1 + p\alpha_2 + q\alpha_3 = \mu,$$

equations (3) and (4) become respectively

$$\left.\begin{aligned} a_1 + pa_2 + qa_3 &= \lambda \\ b_1 + pb_2 + qb_3 &= p\lambda \\ c_1 + pc_2 + qc_3 &= q\lambda \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

and

$$\left.\begin{array}{l} \alpha_1 + p\alpha_2 + q\alpha_3 = \mu \\ \beta_1 + p\beta_2 + q\beta_3 = p\mu \\ \gamma_1 + p\gamma_2 + q\gamma_3 = q\mu \end{array}\right\} \text{................(10).}$$

If (9) and (10) can be satisfied, then (2) can be written in the form

$$\frac{du + p\,dv + q\,dw}{u + pv + qw} = \lambda\, d\phi + \mu\, d\psi,$$

which is immediately integrable in the form

$$u + pv + qw = Ke^{\lambda\phi + \mu\psi}.$$

Eliminating p and q from equations (9) and (10), we should be left with four other relations. Two of these will be

$$\begin{vmatrix} a_1 - \lambda, & a_2\;\;, & a_3 \\ b_1\;\;, & b_2 - \lambda, & b_3 \\ c_1\;\;, & c_2\;\;, & c_3 - \lambda \end{vmatrix} = 0, \quad \begin{vmatrix} \alpha_1 - \mu, & \alpha_2\;\;, & \alpha_3 \\ \beta_1\;\;, & \beta_2 - \mu, & \beta_3 \\ \gamma_1\;\;, & \gamma_2\;\;, & \gamma_3 - \mu \end{vmatrix} = 0.$$

These two conditions require that λ and μ shall be latent roots of m and m' respectively. The other two really reduce to a single one. To demonstrate this we will combine the three equations (9) with each of the three equations (10) successively. We have

$$\begin{aligned} \mu &= \alpha_1 + p\alpha_2 + q\alpha_3 \\ &= A\,\{a_1^2 + b_1a_2 + c_1a_3 + p\,(a_2a_1 + b_2a_2 + c_2a_3) \\ &\quad + q\,(a_3a_1 + b_3a_2 + c_3a_3)\} + B\,(a_1 + pa_2 + qa_3) + C, \end{aligned}$$

by means of the first equations of the three sets (6), (7), (8). Thus we have

$$\begin{aligned} \mu &= A\,\{a_1(a_1 + pa_2 + qa_3) + a_2(b_1 + pb_2 + qb_3) \\ &\qquad + a_3(c_1 + pc_2 + qc_3)\} \\ &\qquad + B(a_1 + pa_2 + qa_3) + C \\ &= A\lambda\,(a_1 + pa_2 + qa_3) + B\lambda + C \\ &= A\lambda^2 + B\lambda + C, \end{aligned}$$

by making repeated use of equations (9). It is also easy to see, from the form of equations (6), (7), (8), that we come to the same result by combining each of the two latter of equations (10) with the three equations (9). Thus the other

two conditions reduce to a single one shewing the manner in which the corresponding values of λ and μ are related. The corresponding values of p and q may be deduced from any two equations out of the sets (9) and (10).

Thus the assumption that (2) may be rendered exact is fully justified, and we have the three exact integrals

$$u + p_1 v + q_1 w = K_1 e^{\lambda_1 \phi + \mu_1 \psi},$$

$$u + p_2 v + q_2 w = K_2 e^{\lambda_2 \phi + \mu_2 \psi},$$

$$u + p_3 v + q_3 w = K_3 e^{\lambda_3 \phi + \mu_3 \psi},$$

from which u, v, w may be determined as functions of ϕ and ψ. From them we see that u, v, w are connected by a single relation not involving ϕ and ψ, viz.

$$(u+p_1 v+q_1 w)^{\lambda_2\mu_3-\lambda_3\mu_2}(u+p_2 v+q_2 w)^{\lambda_3\mu_1-\lambda_1\mu_3}(u+p_3 v+q_3 w)^{\lambda_1\mu_2-\lambda_2\mu_1}$$
$$= K_1^{\lambda_2\mu_3-\lambda_3\mu_2} K_2^{\lambda_3\mu_1-\lambda_1\mu_3} K_3^{\lambda_1\mu_2-\lambda_2\mu_1}.$$

As an illustration of the above, we will take the set of equations

$$du = w d\phi + v d\psi,$$

$$dv = u d\phi + w d\psi,$$

$$dw = v d\phi + u d\psi.$$

The two matrices m and m', with which we are concerned in this case, are respectively

$$\begin{matrix} 0, & 0, & 1 \\ 1, & 0, & 0 \\ 0, & 1, & 0 \end{matrix} \quad \text{and} \quad \begin{matrix} 0, & 1, & 0 \\ 0, & 0, & 1\,. \\ 1, & 0, & 0 \end{matrix}$$

These two matrices are clearly convertible. In fact we have

$$mm' = m'm = 1.$$

Further, we have for the relation connecting the two matrices

$$m' = m^2.$$

Also the characteristic equation of the first matrix is

$$m^3 = 1.$$

Thus, taking ω to denote one of the imaginary cube roots of unity, if we assume

$$\lambda_1 = 1, \ \lambda_2 = \omega, \ \lambda_3 = \omega^2,$$

we have also

$$\mu_1 = 1, \ \mu_2 = \omega^2, \ \mu_3 = \omega.$$

For the equations determining p and q we have

$$p = \lambda, \quad q = p\lambda,$$

and therefore

$$q = \lambda^2 = \mu.$$

Thus, for the three equations determining u, v, w we have

$$u + v + w = K_1 e^{\phi+\psi},$$

$$u + \omega v + \omega^2 w = K_2 e^{\omega\phi+\omega^2\psi},$$

$$u + \omega^2 v + \omega w = K_3 e^{\omega^2\phi+\omega\psi}.$$

The relation connecting u, v, w will therefore be

$$u^3 + v^3 + w^3 - 3uvw = K_1 K_2 K_3.$$

5. We have worked out the simplest case at full length as that is typical of all the rest. In the general case the number of conditions to be satisfied by the coefficients, in order that the system of equations may be exact, is very much in excess of the number of coefficients. It will, however, be found that a large number of them are practically redundant, being direct consequences of the remainder. In fact we shall be left with a large number of disposable constants.

In the case of $2n - 1$ variables, n dependent and $n - 1$ independent, we shall be concerned with $n - 1$ matrices each of n rows and columns. Each pair of these matrices will have to be made convertible. This condition may be secured by selecting a particular matrix of the set and making each of the rest convertible with it. Each of the remaining matrices will therefore be a rational integral function of the selected matrix, of degree $n - 1$. Its expression in terms of the selected matrix will thus involve n constant coefficients. Thus the number of disposable constants will be

$$n^2 + n(n - 2) = 2n(n - 1).$$

As in the special case we have worked out, it will be found that the associated latent roots, that occur in any selected integral, are connected by relations exactly similar to those connecting the matrices from which they are derived. This may be readily established as follows.

Let the selected matrix m be of the form

$$\begin{array}{l} a_{11},\ a_{12},\ a_{13},\ \ldots,\ a_{1n} \\ a_{21},\ a_{22},\ a_{23},\ \ldots,\ a_{2n}. \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1},\ a_{n2},\ a_{n3},\ \ldots,\ a_{nn} \end{array}$$

We will denote another matrix of the set by m', and its elements by the symbol α with subscripts similar to those used in the expression for m. We will also denote the element common to the rth row and sth column of the matrix m^n by the symbol

$$a_{rs}^{(n)}.$$

Then, if the relation connecting the two matrices be of the form

$$m' = A_1 m^{n-1} + A_2 m^{n-2} + \ldots + A_{n-1} m + A_n,$$

we shall have

$$\begin{aligned} \alpha_{11} &= A_1 a_{11}^{(n-1)} + A_2 a_{11}^{(n-2)} + \ldots + A_{n-1} a_{11} + A_n, \\ \alpha_{21} &= A_1 a_{21}^{(n-1)} + A_2 a_{21}^{(n-2)} + \ldots + A_{n-1} a_{21}, \\ \alpha_{31} &= A_1 a_{31}^{(n-1)} + A_2 a_{31}^{(n-2)} + \ldots + A_{n-1} a_{31}, \\ &\&c. \end{aligned}$$

Now, the set of equations analogous to (9) will be

$$\begin{array}{l} a_{11} + p_1 a_{21} + p_2 a_{31} + \ldots + p_{n-1} a_{n1} = \lambda, \\ a_{12} + p_1 a_{22} + p_2 a_{32} + \ldots + p_{n-1} a_{n2} = p_1 \lambda, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{1n} + p_1 a_{2n} + p_2 a_{3n} + \ldots + p_{n-1} a_{nn} = p_{n-1} \lambda, \end{array}$$

from which we immediately deduce that λ is a latent root of m. It will be sufficient for our purpose to make use of the first only out of the set of equations analogous to (10). This will be

$$\mu = \alpha_{11} + p_1 \alpha_{21} + p_2 \alpha_{31} + \ldots + p_{n-1} \alpha_{n1}.$$

Thus, substituting for the a's their values as given above, we have

$$\mu = A_1 \{a_{11}^{(n-1)} + p_1 a_{21}^{(n-1)} + \ldots + p_{n-1} a_{n1}^{(n-1)}\}$$
$$+ A_2 \{a_{11}^{(n-2)} + p_1 a_{21}^{(n-2)} + \ldots + p_{n-1} a_{n1}^{(n-2)}\}$$
$$+ \text{\&c.} + A_n.$$

Now, by the law of multiplication of matrices, we have

$$a_{11}^{(r)} = a_{11} a_{11}^{(r-1)} + a_{12} a_{21}^{(r-1)} + \ldots + a_{1n} a_{n1}^{(r-1)},$$
$$a_{21}^{(r)} = a_{21} a_{11}^{(r-1)} + a_{22} a_{21}^{(r-1)} + \ldots + a_{2n} a_{n1}^{(r-1)},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$a_{n1}^{(r)} = a_{n1} a_{11}^{(r-1)} + a_{n2} a_{21}^{(r-1)} + \ldots + a_{nn} a_{n1}^{(r-1)},$$

and therefore

$$a_{11}^{(r)} + p_1 a_{21}^{(r)} + \ldots + p_{n-1} a_{n1}^{(r)}$$
$$= \lambda \{a_{11}^{(r-1)} + p_1 a_{21}^{(r-1)} + \ldots + p_{n-1} a_{n1}^{(r-1)}\}.$$

By repeated application of this we obtain

$$a_{11}^{(r)} + p_1 a_{21}^{(r)} + \ldots + p_{n-1} a_{n1}^{(r)}$$
$$= \lambda^{r-1} \{a_{11} + p_1 a_{21} + \ldots + p_{n-1} a_{n1}\}$$
$$= \lambda^r.$$

Therefore

$$\mu = A_1 \lambda^{n-1} + A_2 \lambda^{n-2} + \ldots + A_{n-1} \lambda + A_n.$$

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ON THE NUMBER OF PROPER VULGAR FRACTIONS IN THEIR LOWEST TERMS THAT CAN BE FORMED WITH INTEGERS NOT GREATER THAN A GIVEN NUMBER.

By *Professor Sylvester, F.R.S.*

A SLIGHT reflexion will show that the number of such fractions $\left(\frac{1}{1}\right.$ counting as one of them$\left.\right)$ with the limit n is the sum of the totients of all the numbers from 1 to n.

Let us use Ej as usual to denote the integer part of j, τEj to denote the totient (or number of numbers not exceeding and prime) to Ej, and JEj to denote the sum of such totients for all numbers from 1 to j. Then we may establish the following exact equation given by the author of this article, but without proof and with some slight inaccuracy, in the *Phil. Mag.* for April, 1883. The equation is

$$JEj + JE(\tfrac{1}{2}j) + JE(\tfrac{1}{3}j) + \text{etc.},$$

or, more shortly,

$$\Sigma_1^\infty JE\frac{j}{i} = \tfrac{1}{2}\{(Ej)^2 + (Ej)\} \qquad \text{..............(1).}$$

The proof is as follows. Remarking that $E(j-1) = Ej - 1$, the right-hand side of equation (1), when j is reduced to $j-1$ obviously suffers a diminution equal to Ej.

On the left-hand side of the equation any term $JE\frac{j}{i}$ remains unaltered, when for j is written $(j-1)$, unless Ej is divisible by i, in which case the term undergoes a diminution JEj. Thus *ex. gr.* $J\frac{100}{11} - J\frac{99}{11} = 0$, but $J\frac{100}{5} - J\frac{99}{5} = J(20)$. And, as in the case supposed, $\frac{Ej}{i}$ is a factor of Ej, the total diminution, unless one when $j-1$ replaces j, will be the sum of the totients of the factors of Ej, which by a known-theorem equals Ej. Hence equation (1) is satisfied for j if it is satisfied for $j-1$, and as it is true when $Ej = 1$ it is true for all values of j, as was to be proved. From equation (1) it follows that JEj is of the order $(Ej)^2$, and making $JEj = \mu(Ej)^2 + ej$, where ej is zero when $j = \infty$, we obtain

$$\mu\left(1 + \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{16} + \ldots\right) = 1,$$

or
$$\mu = \frac{6}{\pi^2}, \text{ or approximately } Jj = \frac{3j^2}{\pi^2}.$$

In the tables in the *Phil. Mag.* for April and September, 1883, the value of Jj is computed up to $j = 1{,}000$ and compared with the mean value $\frac{3}{\pi^2}j^2$. From this table it appears that Jj is always intermediate between $\frac{3}{\pi^2}j^2$ and $\frac{3}{\pi^2}(j+1)^2$, and much nearer to their mean, which to an insignificant fraction *près* is the same as $\frac{3}{\pi^2}(j^2+j)$ than it is to either extreme. The first, at least, of these statements ought to be susceptible of proof.

As a matter of philosophical interest as embodying a principle applicable to other cases, I will show how I originally found the value $\frac{3}{\pi^2}j^2$ for the number of proper vulgar fractions in their lowest terms that can be formed by means of the first integers.

It is obvious that the probability of any unknown number being divisible by a prime number i is $\frac{1}{i}$, and of any two numbers, being each so divisible, is $\frac{1}{i^2}$, so that the probability of two unknown numbers being each *not* divisible either by 2, 3, 5, 7, n, or any other prime, will be

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)\left(1-\frac{1}{n^2}\right)\ldots,$$

which we know is equal to the sum of the reciprocal of the squares of the natural numbers, *i.e.* is equal to $\frac{6}{\pi^2}$. Hence the number of fractions in their lowest terms that can be got by combining each of j integers with each of i others, found *roughly* by adding together the probable expectation of any such combination consisting of two relative primes, will be $\frac{6}{\pi^2}j^2$, and the number of *proper* fractions in their lowest terms so capable of being formed will be the half of this or $\frac{3j^2}{\pi^2}$. It appears incidentally from this that the average or mean value of the totient of any number is $\frac{3}{\pi^2}$ into, or rather more than, $\frac{3}{10}$ths of that number.

In like manner, if we define a mid-prime to the number $2n$ to be one which is greater than $\frac{1}{2}(n)$ and less than $\frac{1}{2}(3n)$, the *range* of numbers amongst which such primes are to be found will, to a unit *près*, be n. Let us call the number of such mid-primes μ. Then the probability of any number and its complement in respect to $2n$ being each of them primes will be $\frac{\mu^2}{n^2}$. If now we seek the number of solutions of the equation in prime numbers $x+y=2n$, which will be an even or an odd number, according as n is a composite number or a prime, we way suppose a row of n white balls and n black balls, each series being marked with all the numbers from 1 to n inclusive. It follows from what has been said that the sum of the expectation of x being inscribed on any one of the white balls being itself a prime, and its complement $2n-x$ upon one of the black balls being so likewise, will be $n.\frac{\mu^2}{n^2}$, *i.e.* $\frac{\mu^2}{n}$,* and as the same will be true when x is a figure on a black ball and $2n-x$ on a white, the total value of the expectation of the equation in *primes* $x+y=2n$ being satisfied will be the double of this,

* μ is of the order of and ultimately in a ratio of equality with, $\frac{n}{\log n}$ in the sense that, however small ϵ be taken, a limit L_ϵ can be found such for all values of n beyond it, μ will be limited on the two sides by $(1\pm\epsilon)\frac{n}{\log n}$; this follows demonstrably from a known theorem proved within the last few years, and as a consequence we see that the number of solutions in "mid-primes" of the equation $x+y=2n$ will necessarily be of the same order as $\frac{n}{(\log n)^2}$ and *presumably* in a ratio of equality with it in the sense explained above, but this, of course, awaits demonstration.

or $\frac{2\mu^2}{n}$. I have had tables constructed for determining the number of the solutions of this equation (x and y being primes) from $2n = 2$ up to $2n = 500$.

Call the number of solutions for any value of n, $\theta\frac{\mu^2}{n}$; on taking the average value of θ for all values of $2n$ on the 1st, 2nd, 3rd, 4th, 5th, centuries respectively, it will be found that

$$\begin{aligned} \tfrac{1}{2}\theta &= \cdot 96344 \\ &= \cdot 99349 \\ &= 1\cdot 00603 \\ &= \cdot 98281 \\ &= \cdot 99764, \end{aligned}$$

of which the sum is 4·94341 and the average is ·98868, agreeing with wonderful nearness to the rough estimate of solutions being $\frac{2\mu^2}{n}$.

I ought not, however, to suppress the fact that, from another point of view, this number might be expected to eventuate as $\frac{\mu^2}{n}$ instead of $\frac{2\mu^2}{n}$.

In equation (1) we may write $F(j)$ for the sum of the totients of all the numbers not exceeding j, and it then takes the form

$$\phi j = \tfrac{1}{2}\{Ej + (Ej)^2\} = Fj + F(\tfrac{1}{2}j) + F(\tfrac{1}{3}j) + F(\tfrac{1}{4}j) + \text{etc.},$$

which, by the well-known formula of reversion (see *Phil. Mag.*, December, 1884*), gives

$$Fj = \phi j - \phi(\tfrac{1}{2}j) - \phi(\tfrac{1}{3}j) + \phi(\tfrac{1}{5}j) + \phi(\tfrac{1}{6}j) \text{ etc.}$$

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* I do not know whether the annexed important case of reversion has been noticed or not: i being greater than unity, let σ_i denote the sum of the *negative* i^{th} powers of the prime numbers 2, 3, 5, 7, etc., and s_i the *logarithm* of the sum of the negative i^{th} powers of the natural numbers 1, 2, 3, 4, etc. (which, when i is an even integer is a known quantity), then it is easily shown that

$$s_i = \sigma_i + \tfrac{1}{2}\sigma_{2i} + \tfrac{1}{3}\sigma_{3i} + \tfrac{1}{4}\sigma_{4i} + \tfrac{1}{5}\sigma_{5i} + \text{etc.},$$

and therefore by reversion

$$\sigma_i = s_i - \tfrac{1}{2}s_{2i} - \tfrac{1}{3}s_{3i} - \tfrac{1}{5}s_{5i} + \tfrac{1}{6}s_{6i} - \tfrac{1}{7}s_{7i} + \tfrac{1}{10}s_{10i} \text{ etc.}$$

A very general case for reversion arises when $\phi i = \Sigma\frac{1}{n^s}\phi(n^r.i)$. In this last application of the formula $r = 1$, $s = 1$; in the case considered in the text relating to Farey series $r = 0$, $s = -1$.

Thus *ex. gr.* the number of terms in a Farey series with 17 as a limit should be equal to

$$\tfrac{1}{2}(17-8-5-3+2-2+1-1-1+1+1-1)$$

$$+\tfrac{1}{2}(289-64-25-9+4-4+1-1-1+1+1-1)$$

i.e. $\frac{1}{2}(1)+\frac{1}{2}(191)$ or 96, which is right.*

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* And so in general, since by a well-known theorem

$$Ej - E(\tfrac{1}{2}j) - E(\tfrac{1}{3}j) + E(\tfrac{1}{6}j), \text{ etc.}$$

is always equal to unity, so that

$$(Ej)^2 - 2JEj + 1 = \{E(\tfrac{1}{2}j)^2\} + \{E(\tfrac{1}{3}j)^2\} - \{E(\tfrac{1}{6}j)^2\}, \text{ etc.},$$

we have always

$$2JEj - 1 = (Ej)^2 - \{E(\tfrac{1}{2}j)^2\} - \{E(\tfrac{1}{3}j)^2 + \{E(\tfrac{1}{6}j)^2\}, \text{ etc.}$$

a very convenient, and, I believe, new formula for calculating the number of fractions in their lowest terms where neither numerator nor denominator exceeds j.

To this E theorem there exists a pendant which may be called the H theorem, viz. let Hx mean the nearest integer (when there is one) to x, but when x is midway between two integers Hx is to denote the first integer above x; let $p, q, r, \ldots$ be the primes not exceeding the integer n, and call

$$H_n = n - \Sigma H\frac{n}{p} + \Sigma H\frac{n}{pq} - \Sigma H\frac{n}{pqr}, \text{ etc.};$$

then H_n will be the number of primes greater than n and less than $2n$, so that H_n is always greater than zero; and if $\epsilon(x)$ means unity or zero according as x is a prime or not, we shall always have

$$H_n - H_{n-1} = \epsilon(2n-1) - \epsilon(n).$$

I do not know whether this theorem has been previously noticed. It may be obtained by the Eratosthenes sieve process applied to the progression $n+1$, $n+2$, $n+3$, ..., $2n$, replacing therein every prime number by unity.

If not already known, it may be worth while to register the two following additional theorems concerning E_1n and H_1n, by which I mean what E_n and H_n become when the even prime 2 does not count among the primes p, q, r, which are less than n, viz.

$$E_1n = E\left(\frac{n}{2}\right) - \Sigma E\frac{n}{2p} + \Sigma E\frac{n}{2pq}, \text{ etc.} = E\left(\frac{\log n}{\log 2}\right),$$

$$H_1n = H\frac{n}{2} - \Sigma H\frac{n}{2p} + \Sigma H\frac{n}{2pq}, \text{ etc.} \pm 1.$$

[This paper was sent by Professor Sylvester to the editor on Feb. 12th, 1897, with a letter in which he wrote "I could subsequently send you the valuable table referred to in the text, giving the number of solutions of the equation $x+y=2n$ in prime numbers for all values of n up to 500." In subsequent letters he made several slight additions to the paper. He corrected the proof sheets about the end of the month, and then added the first footnote and the last paragraph of the third note. His death took place on March 15th.]

AN EXAMPLE ILLUSTRATIVE OF THE MOLECULAR THEORY OF MAGNETISM.

By *E. G. Gallop, M.A.*

PROFESSOR EWING, in his treatise on *Magnetic Induction in Iron and other Metals*, has extended Weber's molecular theory so as to explain modern observations on the behaviour of bodies magnetized by induction. In his theory the constraint on the molecular magnets is supposed to be due to the magnetic action of the neighbouring molecular magnets. The mathematical treatment of a general system of such magnets is complicated. It is proposed in this paper to consider a very simple arrangement of molecules which illustrates some of the main features of magnetic induction, and at the same time requires but simple calculations for the discussion of its positions of equilibrium when disturbed by the introduction of magnetic force.

§ 1. Consider a pair of magnetic molecules, each of magnetic moment m, having their centres fixed at a distance c apart. Let a magnetic force H be made to act on the molecules in a direction perpendicular to the line joining the centres. It is required to find the positions of equilibrium, and to determine which of these positions are stable.

Let the axes of the molecules make angles θ, ϕ with the line c. The potential energy is therefore given by

$$W = \frac{m^2}{c^3}\left[\cos(\theta - \phi) - 3\cos\theta\cos\phi\right] - Hm(\sin\theta + \sin\phi),$$

if we suppose the magnets to lie in a plane parallel to H.

The positions of stable equilibrium will be found by making W a minimum for variations of θ and ϕ. For convenience, write

$$H = \frac{m}{c^3}h, \quad W = \frac{m^2}{c^3}w,$$

so that

$$w = \sin\theta\sin\phi - 2\cos\theta\cos\phi - h(\sin\theta + \sin\phi),$$

and w is to be a minimum.

We have

$$\frac{dw}{d\theta} = \cos\theta\sin\phi + 2\sin\theta\cos\phi - h\cos\theta = 0 \ldots (1),$$

$$\frac{dw}{d\phi} = \sin\theta\cos\phi + 2\cos\theta\sin\phi - h\cos\phi = 0 \ldots (2),$$

$$\frac{d^2w}{d\theta^2} = -\sin\theta\sin\phi + 2\cos\theta\cos\phi + h\sin\theta \ldots\ldots (3),$$

$$\frac{d^2w}{d\phi^2} = -\sin\theta\sin\phi + 2\cos\theta\cos\phi + h\sin\phi \ldots\ldots (4),$$

$$\frac{d^2w}{d\theta\, d\phi} = \cos\theta\cos\phi - 2\sin\theta\sin\phi \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

[It will be noticed that if $H = 0$, the positions of stable equilibrium are those in which the magnets point in the same direction along the line c; that is, $\theta = \phi = 0$, or $\theta = \phi = \pi$. The other positions of equilibrium are unstable.]

From equations (1), (2) by subtraction

$$\sin(\theta - \phi) = h(\cos\theta - \cos\phi);$$

therefore

$$2\sin\tfrac{1}{2}(\theta-\phi)\cos\tfrac{1}{2}(\theta-\phi) = -2h\sin\tfrac{1}{2}(\theta-\phi)\sin\tfrac{1}{2}(\theta+\phi);$$

therefore, either $\sin\frac{1}{2}(\theta-\phi) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6),$

or $\cos\frac{1}{2}(\theta-\phi) = -h\sin\frac{1}{2}(\theta+\phi) \ldots\ldots\ldots\ldots (7).$

Let us first suppose (6) to hold, so that $\theta - \phi = 2n\pi$, and both magnets point in the same direction.

From (1) we have

$$3\sin\theta\cos\theta = h\cos\theta,$$

and therefore $\cos\theta = 0$, or $\sin\theta = \frac{1}{3}h$.

The magnets must therefore either be at right angles to c or inclined to c at an angle $\sin^{-1}(\frac{1}{3}h)$.

The second position is possible only if $h < 3$.

For the inclined position, we find by substitution in (3), (4), (5)

$$\frac{d^2w}{d\theta^2} = \frac{d^2w}{d\phi^2} = 2,$$

$$\frac{d^2w}{d\theta\, d\phi} = 1 - \tfrac{1}{3}h^2.$$

For either maximum or minimum of w it is necessary that

$$\frac{d^2w}{d\theta^2}\,\frac{d^2w}{d\phi^2} > \left(\frac{d^2w}{d\theta\,d\phi}\right)^2,$$

or

$$2^2 > (1 - \tfrac{1}{3}h^2)^2;$$

that is,

$$(1 + \tfrac{1}{3}h^2)(3 - \tfrac{1}{3}h^2) > 0,$$

which will be satisfied by $h < 3$.

Also $\dfrac{d^2w}{d\theta^2}$ is positive. Therefore the inclined position, when possible, is stable.

Next consider the position perpendicular to c. In this case

$$\frac{d^2w}{d\theta^2}\,\frac{d^2w}{d\phi^2} - \left(\frac{d^2w}{d\theta\,d\phi}\right)^2 = (h-1)^2 - 2^2$$
$$= (h-3)(h+1),$$

which is positive if $h > 3$, but negative if $h < 3$. Moreover $\dfrac{d^2w}{d\theta^2}$ is positive when $h > 1$. Hence the position $\theta = \phi = \frac{1}{2}\pi$ is stable when $h > 3$, unstable when $h < 3$. The position $\theta = \phi = -\frac{1}{2}\pi$ is always unstable when h is positive.

Next, suppose equation (7) to hold. Add (1) and (2). Then

$$3\sin(\theta+\phi) = h(\cos\theta + \cos\phi)$$
$$= 2h\cos\tfrac{1}{2}(\theta+\phi)\cos\tfrac{1}{2}(\theta-\phi)$$
$$= -2h^2\cos\tfrac{1}{2}(\theta+\phi)\sin\tfrac{1}{2}(\theta+\phi)$$
$$= -h^2\sin(\theta+\phi).$$

Therefore $\sin(\theta+\phi) = 0$ and $\theta + \phi = n\pi$.

The new positions are easily found from this equation and (7); but it will be discovered on examination that they are all unstable.

Now let H increase from zero. The magnets will at first point in the same direction along the line c. As H increases the magnets will turn towards the direction of H, so as to make the angle θ with c such that

$$\sin\theta = \tfrac{1}{3}h = \tfrac{1}{3}\frac{Hc^3}{m}.$$

When h becomes equal to 3 the magnets will point in the direction of H perpendicular to the line c. If H is still further increased the magnets will continue unmoved in this position.

If H is diminished to zero the magnets will return to their original positions, provided we suppose that there is some slight circumstance favouring the return to the original position rather than to the exactly opposite position. If H is reversed a similar series of positions will be taken up on the other side of c.

§ 2. Next suppose H to act parallel to the line c, opposite to the original direction in which the axes pointed. Then

$$w = \sin\theta\sin\phi - 2\cos\theta\cos\phi + h(\cos\theta + \cos\phi).$$

For equilibrium

$$\cos\theta\sin\phi + 2\sin\theta\cos\phi = h\sin\theta,$$

$$\sin\theta\cos\phi + 2\cos\theta\sin\phi = h\sin\phi.$$

By subtraction

$$\sin(\theta - \phi) = 2h\sin\tfrac{1}{2}(\theta - \phi)\cos\tfrac{1}{2}(\theta + \phi),$$

and therefore

$$\theta - \phi = 2n\pi \text{ or } \cos\tfrac{1}{2}(\theta - \phi) = h\cos\tfrac{1}{2}(\theta + \phi).$$

Taking the first alternative, we have

$$3\sin\theta\cos\theta = h\sin\theta,$$

and therefore $\theta = 0,\ \pi,\ \text{or}\ \cos^{-1}(\tfrac{1}{3}h).$

It will be found that, when $\theta = 0$,

$$\frac{d^2w}{d\theta^2} = 2 - h = \frac{d^2w}{d\phi^2}, \qquad \frac{d^2w}{d\theta\, d\phi} = 1,$$

and therefore for stability

$$(2 - h)^2 > 1 \text{ or } (1 - h)(3 - h) > 0.$$

The original position will therefore be stable if $h < 1$, unstable if $h > 1$.

The position $\theta = \phi = \pi$ is obviously stable.

The inclined position and the new positions obtained from the equation

$$\cos\tfrac{1}{2}(\theta - \phi) = h\cos\tfrac{1}{2}(\theta + \phi)$$

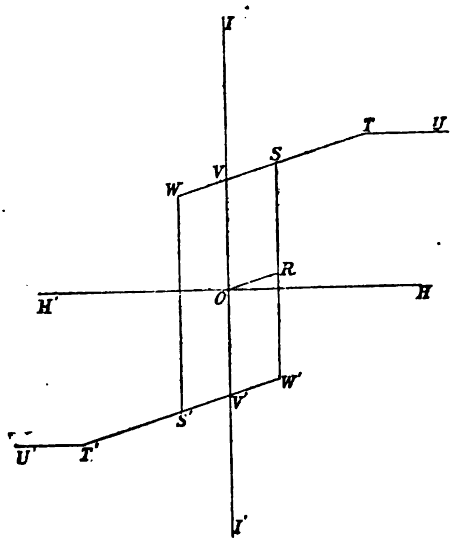
will be found to be unstable.

Hence, if a force H acts along the line c in the direction in which the axes point, the position will be one of stable equilibrium. If H is reversed, the equilibrium will be stable so long as $H < m/c^3$. But if H exceeds this limit the equilibrium

will be unstable; the magnets will turn over and point in the direction of H.

§ 3. Now consider a body made up of four such magnet pairs, A, B, C, D. Let the lines of centres for A and B point north and south; for C and D, east and west. Suppose that initially the magnets of A point north, B south, C east, D west. It will be assumed that each pair is so far from the other three pairs as to be practically uninfluenced by them. It is to be understood that the points of the compass are introduced merely for purposes of description; no account is taken of the earth's magnetic force. Let I be the magnetic moment of the body; then initially $I=0$. Let now a magnetic force H act towards the east; the body will then acquire magnetic moment. Let a point P trace out a diagram indicating the relation between I and H. Let the I axis be vertical and the H axis horizontal on this diagram.

As long as $H<\frac{m}{c^3}$, the C, D pairs will be unaltered whilst the A, B pairs will be deflected through an angle θ, where $\sin\theta=\frac{1}{3}\frac{c^3}{m}H$. Hence, when $H<\frac{m}{c^3}$, $I=4m\sin\theta=\frac{4}{3}c^3H$. The diagram is a straight line OR through the origin.



When H passes $\frac{m}{c^3}$, the D pair turns over and points east. There is a sudden increase in I equal to $4m$. We therefore have for the diagram a vertical line RS of length $4m$. When

$$\frac{m}{c^3} < H < \frac{3m}{c^3}, \quad I = 4m + 4m \sin\theta$$

or

$$I = 4m + \tfrac{4}{3}c^3H.$$

This part of the diagram is a straight line ST, parallel to OR.

When $H > \frac{3m}{c^3}$ all the magnets point east, and $I = 8m$.

The diagram is a horizontal line TU. When H is diminished to $\frac{3m}{c^3}$, the line TU is retraced. As H is diminished to zero, the pairs C, D remain pointing east, but A, B return to their original directions pointing north and south. In this part of the operations

$$I = 4m + \tfrac{4}{3}c^3H,$$

and when H is zero there is residual magnetism $I = 4m$. The diagram is a straight line TV parallel to OR cutting OI in V, where $OV = 4m$.

If now H is reversed, and is therefore considered to be negative, C, D remain pointing east, but A, B are deflected towards the west, so that

$$I = 4m + \tfrac{4}{3}c^3H,$$

till, when $H = -\frac{m}{c^3}$, the eastward direction for C, D becomes unstable; so that when H lies between $-\frac{m}{c^3}$ and $-\frac{3m}{c^3}$, $I = -4m + \frac{4}{3}c^3H$. When H passes $-\frac{3m}{c^3}$, all the magnets point west, and $I = -8m$. The diagram consists of VW parallel to OR, WS' vertical and equal to $8m$, $S'T'$ parallel to OR, and $T'U'$ horizontal. As H is diminished again to zero, the diagram consists of $U'T'S'$ retraced, a straight line $S'V'$ parallel to OR, cutting OI at V', where $OV' = -4m$. If H is increased to $+\frac{m}{c^3}$, the diagram consists of $V'W'$ parallel to OR, and a vertical line $W'RS$ caused by the reversal of the C, D magnets. After S has been reached the figure is retraced.

All the striking features of the diagrams for actual bodies occur in this diagram. We have the first slow increase of magnetic moment represented by OR, the rapid increase represented by RS, the tendency to a limit represented by STU, the residual magnetism OV when the magnetic force is removed, and the general effects of hysteresis represented by the rest of the curve. If instead of four pairs of magnets we had a large number with their lines of centres variously inclined to the direction of H, we should obtain a curve of the same general type, but with rounded corners at S, T, W, S', T', W', agreeing in general outline with the curves for actual bodies.

§ 4. The example also illustrates the loss of energy due to hysteresis. If the body is carried through a cycle of states so that the representative point describes a closed line of which the perimeter of the parallelogram $SWS'W'$ forms part, there will be a loss of energy represented by the area of this closed line, that is, by the area of the parallelogram which is equal to $8m \cdot \frac{2m}{c^3} = \frac{16m^2}{c^3}$. This loss of energy is easily accounted for. In the description of the cycle there are two reversals of the pairs of magnets C, D, taking place when $H = \pm m/c^3$. The difference of the potential energies in the two positions of equilibrium is $4.2m \cdot \frac{m}{c^3} = \frac{8m^2}{c^3}$. In order that the magnets may settle down in their new positions, it is necessary to suppose some kind of frictional resistance acting upon them, such as friction at the pivots. This potential energy lost will therefore appear as heat, having a mechanical equivalent $8m^2/c^3$. The two reversals give a total $16m^2/c^3$, which, as it ought to be, is equal to the area of the parallelogram $SWS'W'$.

SOME SIMPLE PROPERTIES OF DIVISIBILITY.

By Prof. *E. B. Elliott*,

1. There is a pretty theorem, proved by Mr. Segar (*Messenger*, Vol XXII., p. 59), and used by Cayley (Vol. XXII., p. 186) in giving a proof of Hermite's deduction from a theorem of Eisenstein's that n^{r-1} times*

$$m \,.\, m-n \,.\, m-2n \ldots m-(r-1)n$$

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* Cayley's statement is as to n^r times only, but it will readily be seen that his proof really affords Hermite's complete fact.

is divisible by $r\,!$ This theorem is

(I.) *The product of the differences of any n unequal numbers, positive or not, is divisible by the product of the differences of the n numbers* $0, 1, 2, 3, \ldots, n-1$.

That this is an immediate consequence of the fact that the product of r consecutive numbers is divisible by $r\,!$ may be seen as follows. The product of the differences of the n numbers $a_1, a_2, \ldots, a_n$ may be written as a determinant whose first column is

$$\begin{array}{l} 1 \\ a_1 \\ a_1 . a_1 - 1 \\ a_1 . a_1 - 1 . a_1 - 2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_1 . a_1 - 1 . a_1 - 2 \ldots a_1 - (n-2), \end{array}$$

and whose other $n-1$ columns are the results of replacing in this one the suffix 1 by $2, 3, \ldots, n$ respectively. Now in this determinant the constituents in any row have a common divisor, the divisors for the second, third, fourth, ..., n^{th} rows being respectively

$$\begin{array}{l} 1 \quad i.e. \quad 1-0, \\ 2\,! \quad i.e. \quad 2-0.2-1, \\ 3\,! \quad i.e. \quad 3-0.3-1.3-2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ (n-1)\,! \quad i.e. \quad (n-1)-0.(n-1)-1\ldots(n-1)-(n-2). \end{array}$$

The determinant is then divisible by the product of these products; which is the theorem (I.)

2. This method can clearly be utilized to obtain numerous other theorems as to divisibility. Two of these theorems, which admit of expression having like elegance with Mr. Segar's, will here be proved. They are:

(II.) *The product of the differences of any n different square numbers is divisible by the product of the differences of the first n square numbers* $0^2, 1^2, 2^2, \ldots, (n-1)^2$.

(III.) *The product of the differences of the squares of any n different odd numbers, multiplied by the product of the n odd numbers themselves, is divisible by the product of the differences of the squares of the first n odd numbers* 1, 3, 5, ..., $2n-1$, *multiplied by the product of* 1, 3, 5, ..., $2n-1$.

3. The one lemma necessary is that the product of any n consecutive numbers and of the arithmetic mean of these numbers is divisible by the product of 1, 2, 3, ..., n and of their arithmetic mean. This is clear, for the first product is half the sum of two products of $n+1$ consecutive numbers, and the second product is half of $(n+1)$!

For n odd $(=2r-1)$ the fact thus expressed is that, a being any number,

$$\frac{a^2 . a^2-1^2 . a^2-2^2 \ldots a^2-(r-1)^2}{r^2 . r^2-1^2 . r^2-2^2 \ldots r^2-(r-1)^2}$$

is integral. For n even $(=2r)$ it is that, a being any number,

$$\frac{2a+1 . (2a+1)^2-1^2 . (2a+1)^2-3^2 \ldots (2a+1)^2-(2r-1)^2}{2r+1 . (2r+1)^2-1^2 . (2r+1)^2-3^2 \ldots (2r+1)^2-(2r-1)^2}$$

is integral. In the latter we have multiplied numerator and denominator by 2^{2r+1}.

Notice that the two facts are that the coefficients in the expansions in terms of $2\sin\theta$ of $2\cos 2a\theta$ and $2\sin(2a+1)\theta$ respectively are all integral.

4. Theorem (II.) follows from the first case of the lemma of the last article. The product of the differences of the squares $a_1^2, a_2^2, \ldots, a_n^2$ may be written as a determinant in which the first column is

$$\begin{array}{l}
1 \\
a_1^2 \\
a_1^2 . a_1^2-1^2 \\
a_1^2 . a_1^2-1^2 . a_1^2-2^2 \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
a_1^2 . a_1^2-1^2 . a_1^2-2^2 \ldots a_1^2-(n-2)^2,
\end{array}$$

and the other $n-1$ columns are the results of replacing in this one the suffix 1 by 2, 3, ..., n respectively. By the lemma all constituents in the second, third, fourth, ..., n^{th} rows have the respective common divisors

$$1^2-0^2$$

$$2^2-0^2.2^2-1^2$$

$$3^2-0^2.3^2-1^2.3^2-2^2$$

....

$$(n-1)^2-0^2.(n-1)^2-1^2.(n-1)^2-2^2...(n-1)^2-(n-2)^2.$$

The determinant is then divisible by the product of these products, *i.e.* by the product of the differences of the n squares $0^2, 1^2, 2^2, ..., (n-1)^2$.

5. Theorem (III.) follows from the second case of the lemma of § 3. The product of the differences of $(2a_1+1)^2$, $(2a_2+1)^2$, ... $(2a_n+1)^2$, multiplied by the product

$$2a_1+1.2a_2+1...2a_n+1,$$

may be written as a determinant whose first column is

$$2a_1+1$$

$$2a_1+1.(2a_1+1)^2-1^2$$

$$2a_1+1.(2a_1+1)^2-1^2.(2a_1+1)^2-3^2$$

....

....

$$2a_1+1.(2a_1+1)^2-1^2.(2a_1+1)^2-3^2...(2a_1+1)^2-(2n-3)^2,$$

the remaining columns being of like form with the other $n-1$ suffixes respectively. By the lemma the constituents of any row here have a common divisor, the common divisors for the several rows being

$$1$$

$$3.3^2-1^2$$

$$5.5^2-1^2.5^2-3^2$$

....

....

$$2n-1.(2n-1)^2-1^2.(2n-1)^2-3^2...(2n-1)^2-(2n-3)^2,$$

the determinant is then divisible by the product of all these products, *i.e.* by the product contemplated in the theorem.

A NEW PROOF OF PICARD'S THEOREM.

By *E. W. Barnes, B.A.*, Trinity College, Cambridge.

It may be of interest to give the following proof of Picard's theorem, which is practically the same as an algebraical proof given by Professor Forsyth in his lectures on Linear Differential Equations, October Term, 1895, except that I use Cayley's notation of matrices by which the algebraical manipulation is reduced to that of an equation of the first order.

The theorem to be proved is "Assuming that the integrals of the linear differential equation with doubly periodic coefficients

$$\frac{d^u y}{du^u} + p_1 \frac{d^{u-1} y}{du^{u-1}} + \ldots + p_u y = 0$$

are uniform, then at least one is a doubly periodic function of the second kind."

Let $\varpi_1(x) \ldots \varpi_2(x)$ be any complete system of integrals—which we denote by the row-letter $\varpi(x)$. Then, ω and ω being the periods of the coefficients $p_1 \ldots p_u$, we have

$$\varpi_1(x+\omega), \ldots, \varpi_u(x+\omega),$$

and
$$\varpi_1(x+\omega'), \ldots, \varpi_u(x+\omega').$$

also solutions of the equation, so that

$$\varpi(x+\omega) = a\varpi(x),$$

and
$$\varpi(x+\omega') = b\varpi(x),$$

where a and b are matrices, each of u rows and columns.

Since
$$\varpi(x+\omega+\omega') = \varpi(x+\omega'+\omega),$$

we have at once
$$ab = ba.$$

Suppose now that $F(x)$ is an integral of the equation so chosen that

$$F(x+\omega) = cF(x),$$

c being a constant.

CONTENTS.

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The series $1 - \frac{1}{3^{2n}} + \frac{1}{5^{2n}} - \&c.$, *and* $1 + \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} + \&c.$, §§ 95–98.

§ 95. The results in § 90 afford interesting expressions for the series

$$1 - \frac{1}{3^{2n}} + \frac{1}{5^{2n}} - \frac{1}{7^{2n}} + \&c.,$$

and
$$1 + \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} + \frac{1}{7^{2n+1}} + \&c.,$$

as definite integrals involving the Bernoullian function: viz. we have

$$(2n-1)!\left\{1 - \frac{1}{3^{2n}} + \frac{1}{5^{2n}} - \&c.\right\} = (-1)^n \pi^{2n} \int_0^{\frac{1}{2}} \frac{A'_{2n}(x)}{\cos \pi x}\,dx,$$

and

$$(2n)!\left\{1 + \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} + \&c.\right\} = (-1)^n \pi^{2n+1} \int_0^{\frac{1}{2}} \frac{A'_{2n+1}(x)}{\sin \pi x}\,dx$$

$$= (-1)^{n+1} (2\pi)^{2n+1} \int_0^{\frac{1}{2}} \frac{A_{2n+1}(x)}{\sin 2\pi x}\,dx.$$

As examples, putting $n = 1$, these formulæ become

$$1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \&c. = -\pi^2 \int_0^{\frac{1}{2}} \frac{\frac{1}{2}x - \frac{1}{4}}{\cos \pi x}\,dx,$$

and

$$2\left\{1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + \&c.\right\} = -\pi^3 \int_0^{\frac{1}{2}} \frac{\frac{1}{2}x^2 - \frac{1}{2}x}{\sin \pi x}\,dx$$

$$= 8\pi^3 \int_0^{\frac{1}{2}} \frac{\frac{1}{6}x^3 - \frac{1}{2}x^2 + \frac{1}{6}x}{\sin 2\pi x}\,dx.$$

§ 96. The integrals may be put in a rather more convenient form by replacing x by $\frac{1}{2} - x$ and using the formulæ (§ 70)

$$A_n'(\tfrac{1}{2} - x) = (-1)^{n-1} A_n(\tfrac{1}{2} + x),$$

$$A_n(\tfrac{1}{2} - x) = (-1)^n A_n(\tfrac{1}{2} + x).$$

We thus find

$$(2n-1)!\left\{1-\frac{1}{3^{2n}}+\frac{1}{5^{2n}}-\&c.\right\}=(-1)^{n-1}\pi^{2n}\int_0^{\frac{1}{2}}\frac{A'_{2n}(x+\frac{1}{2})}{\sin\pi x}\,dx,$$

and

$$(2n)!\left\{1+\frac{1}{3^{2n+1}}+\frac{1}{5^{2n+1}}+\&c.\right\}=(-1)^{n-1}\pi^{2n+1}\int_0^{\frac{1}{2}}\frac{A'_{2n+1}(x+\frac{1}{2})}{\cos\pi x}\,dx$$

$$=(-1)^n(2\pi)^{2n+1}\int_0^{\frac{1}{2}}\frac{A_{2n+1}(x+\frac{1}{2})}{\sin 2\pi x}\,dx.$$

§ 97. If we put

$$A_n'(x+\tfrac{1}{2})=\alpha_n'(x),$$
$$A_n\,(x+\tfrac{1}{2})=\alpha_n\,(x),$$

and denote the series

$$1-\frac{1}{3^{2n}}+\frac{1}{5^{2n}}-\&c.,\quad 1+\frac{1}{3^{2n+1}}+\frac{1}{5^{2n+1}}+\&c.$$

by u_{2n}, v_{2n+1} respectively, these formulæ may be written

$$(2n-1)!\,u_{2n}=(-1)^{n-1}\pi^{2n}\int_0^{\frac{1}{2}}\frac{\alpha'_{2n}(x)}{\sin\pi x}\,dx,$$

$$(2n)!\,v_{2n+1}=(-1)^{n-1}\pi^{2n+1}\int_0^{\frac{1}{2}}\frac{\alpha'_{2n+1}(x)}{\cos\pi x}\,dx$$

$$=(-1)^n(2\pi)^{2n+1}\int_0^{\frac{1}{2}}\frac{\alpha_{2n+1}(x)}{\sin 2\pi x}\,dx.$$

§ 98. The calculation of the series u_{2n} has been considered in Vol. XXIII. of the *Messenger*, where references are given to other papers relating to the same subject.* The value of u_2 has been calculated to twenty places of decimals.†

Putting $n=1$ the first formula gives

$$u_2=\pi^2\int_0^{\frac{1}{2}}\frac{\frac{1}{2}x}{\sin\pi x}\,dx,$$

---

* "On series involving inverse even powers of subeven and supereven numbers," pp. 176–184.

† *Proc. Lond. Math. Soc.*, Vol. VIII., p. 203.

which agrees with the known result

$$\int_0^{\frac{1}{2}\pi} \frac{x}{\sin x}\, dx = 2u_2.\text{*}$$

The functions $\alpha_n'(x)$ and $\alpha_n(x)$, §§ 99–102.

§ 99. The function $\alpha_n'(x)$ is expressed in powers of x by the formula

$$2^n \alpha_n'(x) = (2x)^{n-1} - (n-1)_2 E_1 (2x)^{n-3}$$
$$+ (n-1)_4 E_2 (2x)^{n-5} - \dots \;(Q.J.\ \S\, 178,\ \text{p. } 95),$$

the series being continued so long as the exponents of $2x$ are not negative, and $E_1, E_2, E_3, \dots$ being the Eulerian numbers numbers 1, 5, 61, ...

Thus we have

$$\alpha_1'(x) = \tfrac{1}{2},$$
$$\alpha_2'(x) = \tfrac{1}{2}x,$$
$$\alpha_3'(x) = \tfrac{1}{2}x^2 - \tfrac{1}{8},$$
$$\alpha_4'(x) = \tfrac{1}{2}x^3 - \tfrac{3}{8}x,$$
$$\alpha_5'(x) = \tfrac{1}{2}x^4 - \tfrac{3}{4}x^2 + \tfrac{5}{32},$$
$$\alpha_6'(x) = \tfrac{1}{2}x^5 - \tfrac{5}{4}x^3 + \tfrac{25}{32}x,$$
$$\alpha_7'(x) = \tfrac{1}{2}x^6 - \tfrac{15}{8}x^4 + \tfrac{75}{32}x^2 - \tfrac{61}{128},$$
$$\alpha_8'(x) = \tfrac{1}{2}x^7 - \tfrac{21}{8}x^5 + \tfrac{175}{32}x^3 - \tfrac{427}{128}x.$$

As in the case of the A'-functions (§ 70), we have the formula

$$\frac{d}{dx}\alpha_n'(x) = (n-1)\,\alpha'_{n-1}(x);$$

and we deduce also from the A'-formulæ (§§ 55 and 70) that

$$\alpha'_{2n}(0) = 0,$$
$$2^{2n+1}\alpha'_{2n+1}(0) = (-1)^n E_n,$$

and

$$\alpha'_{2n}(\tfrac{1}{2}) = (-1)^{n-1}(2^{2n}-1)\frac{B_n}{2n},$$
$$\alpha_1'(\tfrac{1}{2}) = \tfrac{1}{2},\quad \alpha'_{2n+1}(\tfrac{1}{2}) = 0,\quad n > 0.$$

---

* *Messenger*, Vol. VI., p. 76.

§ 100. From the formulæ (§ 2),

$$\tfrac{1}{2}a\frac{\cos xa}{\cos\frac{1}{2}a} = a\alpha_1'(x) - \frac{a^3}{2!}\alpha_3'(x) + \frac{a^5}{4!}\alpha_5'(x) - \&c.,$$

$$\tfrac{1}{2}a\frac{\sin xa}{\cos\frac{1}{2}a} = a^2\alpha_2'(x) - \frac{a^4}{3!}\alpha_4'(x) + \frac{a^6}{5!}\alpha_6'(x) - \&c.,$$

we find, by multiplying by $\cos\frac{1}{2}a$, and equating coefficients,

$$2^n\alpha_n'(x) + (n-1)_2 2^{n-2}\alpha'_{n-2}(x) + (n-1)_4 2^{n-4}\alpha'_{n-4}(x) + \ldots = 2^{n-1}x^{n-1},$$

the series being continued as far as the term $(n-1)\,2^2\alpha_2'(x)$ or $2\alpha_1\,(x)$ according as n is even or uneven.

§ 101. The function $\alpha_n(x)$ is expressed in powers of x by the formula

$$n\alpha_n(x) = x^n - (n)_2\frac{(2-1)B_1}{2}x^{n-2}$$
$$+ (n)_4\frac{(2^3-1)B_2}{2^3}x^{n-4} - \ldots \quad (Q.J.\ \S\S\ 20,\ 21,\ \text{p. } 13),$$

the series being continued as far as the term involving x^0 or x according as n is even or uneven.

We thus find

$$\alpha_1(x) = x,$$
$$\alpha_2(x) = \tfrac{1}{2}x^2 - \tfrac{1}{24},$$
$$\alpha_3(x) = \tfrac{1}{3}x^3 - \tfrac{1}{12}x,$$
$$\alpha_4(x) = \tfrac{1}{4}x^4 - \tfrac{1}{8}x^2 + \tfrac{7}{960},$$
$$\&c., \qquad \&c.$$

We have also

$$\frac{d}{dx}\alpha_n(x) = (n-1)\,\alpha_{n-1}(x),$$

and

$$2^{2n}\alpha_{2n}(0) = (-1)^n(2^{2n}-2)\frac{B_n}{2n},$$

$$\alpha_{2n+1}(0) = 0,$$

$$\alpha_{2n}(\tfrac{1}{2}) = (-1)^{n-1}\frac{B_n}{2n},$$

$$\alpha_1(\tfrac{1}{2}) = \tfrac{1}{2}, \quad \alpha_{2n+1}(\tfrac{1}{2}) = 0,\ n > 0.$$

§ 102. From the formulæ (§ 8)

$$\tfrac{1}{2}a\,\frac{\sin xa}{\sin\frac{1}{2}a} = a\alpha_1(x) - \frac{a^3}{2\,!}\alpha_3(x) + \frac{a^5}{4\,!}\alpha_5(x) - \&c.,$$

$$\tfrac{1}{2}a\,\frac{\cos xa}{\sin\frac{1}{2}a} = 1 - a^2\alpha_2(x) + \frac{a^4}{3\,!}\alpha_4(x) - \&c.,$$

we find, by multiplying by $\sin\frac{1}{2}a$, and equating coefficients,

$$n.2^n\alpha_n(x) + (n)_3\,2^{n-2}\alpha_{n-2}(x) + (n)_5\,2^{n-4}\alpha_{n-4}(x) + \ldots = 2^n x^n,$$

the last term being $2\alpha_1(x)$ when n is uneven, and the last two terms being $n.2^2\alpha_2(x) + \dfrac{1}{n+1}$ when n is even.

For example, putting $n = 1, 2, \ldots,$

$$2\alpha_1(x) = 2x,$$

$$2.2^2\alpha_2(x) + \tfrac{1}{3} = 2^2x^2,$$

$$3.2^3\alpha_3(x) + 2\alpha_1(x) = 2^3x^3,$$

$$4.2^4\alpha_4(x) + 4.2^2\alpha_2(x) + \tfrac{1}{5} = 2^4x^4,$$

&c., &c.

The integrals $\displaystyle\int_0^{\frac{1}{2}\pi}\frac{x^n}{\sin x}\,dx$, &c., §§ 103–105.

§ 103. From § 89, putting $n = 1$ and 2,

$$u_2 = \pi^2\int_0^{\frac{1}{2}}\frac{\frac{1}{2}x}{\sin\pi x}\,dx,$$

$$3\,!\,u_4 = -\pi^4\int_0^{\frac{1}{2}}\frac{\frac{1}{2}x^3 - \frac{3}{8}x}{\sin\pi x}\,dx,$$

whence we deduce

$$\int_0^{\frac{1}{2}}\frac{x^3}{\sin\pi x}\,dx = \tfrac{3}{2}\frac{u_2}{\pi^2} - 12\frac{u_4}{\pi^4};$$

or, replacing πx by x under the integral sign,

$$\int_0^{\frac{1}{2}\pi}\frac{x^3}{\sin x}\,dx = \tfrac{3}{2}\pi^2u_2 - 12u_4.$$

In a similar manner we may obtain the value of

$$\int_0^{\frac{1}{2}\pi} \frac{x^5}{\sin x}\,dx,$$

in terms of u_2, u_4, u_6; and so on.

§ 104. The general formula giving

$$\int_0^{\frac{1}{2}\pi} \frac{x^{2n-1}}{\sin x}\,dx$$

in terms of u_2, u_4, ..., u_{2n} may be easily obtained by means of the relation (§ 100)

$$2^{2n}\alpha'_{2n}(x) + (2n-1)_2\, 2^{2n-2}\alpha'_{2n-2}(x) + \ldots$$
$$+ (2n-1)\, 2^2\alpha_2'(x) = 2^{2n-1}x^{2n-1}.$$

We thus find

$$2^{2n}u_{2n} - \frac{\pi^2.2^{2n-2}u_{2n-2}}{2\,!} + \frac{\pi^4.2^{2n-4}u_{2n-4}}{4\,!} - \ldots + (-1)^{n-1}\frac{\pi^{2n-2}.2^2u_2}{(2n-2)\,!}$$
$$= (-1)^{n-1}\frac{\pi^{2n}.2^{2n-1}}{(2n-1)\,!}\int_0^{\frac{1}{2}} \frac{x^{2n-1}}{\sin \pi x}\,dx;$$

or, replacing πx by x under the integral sign, and writing the terms of the series in the reverse order,

$$\frac{2^{2n-1}}{(2n-1)\,!}\int_0^{\frac{1}{2}\pi} \frac{x^{2n-1}}{\sin x}\,dx = \frac{\pi^{2n-2}2^2u_2}{(2n-2)\,!} - \frac{\pi^{2n-4}2^4u_4}{(2n-4)\,!} + \ldots + (-1)^{n-1}2^{2n}u_{2n}.$$

For example, putting $n = 3$,

$$\int_0^{\frac{1}{2}\pi} \frac{x^5}{\sin x}\,dx = \tfrac{5}{8}\pi^4u_2 - 30\pi^2u_4 + 240u_6.$$

§ 105. Similarly by means of the formula (§ 100),

$$2^{2n+1}\alpha'_{2n+1}(x) + (2n)_2\, 2^{2n-1}\alpha'_{2n-1}(x) + \ldots + (2n)_2\, 2^3\alpha_3'(x) = (2x)^{2n} - 1,$$

we may obtain the value of

$$\int_0^{\frac{1}{2}} \frac{(2x)^{2n} - 1}{\cos \pi x}\,dx$$

in terms of v_3, v_5, ..., v_{2n+1}.

Evaluation of other integrals, §§ 106–109.

§ 106. Since

$$\pi\int_0^x \frac{dx}{\cosh 2\pi t - \cos 2\pi x} = \frac{1}{\sinh 2\pi t}\tan^{-1}\left(\frac{\tan \pi x}{\tanh \pi t}\right),$$

we may obtain other formulæ by integrating with regard to x the results (§§ 4 and 9),

$$\int_0^\infty \frac{t^{2n}\cosh \pi t\, dt}{\cosh 2\pi t - \cos 2\pi x} = (-1)^n \frac{A'_{2n+1}(x)}{2\sin \pi x},$$

$$\int_0^\infty \frac{t^{2n-1}\sinh \pi t\, dt}{\cosh 2\pi t - \cos 2\pi x} = (-1)^n \frac{A'_{2n}(x)}{2\cos \pi x},$$

$$\int_0^\infty \frac{t^{2n}dt}{\cosh 2\pi t - \cos 2\pi x} = (-1)^{n+1} \frac{A_{2n+1}(x)}{\sin 2\pi x}.$$

We thus find

$$\int_0^\infty \tan^{-1}\left(\frac{\tan \pi x}{\tanh \pi t}\right)\frac{t^{2n}dt}{\sinh \pi t} = (-1)^n \pi \int_0^x \frac{A'_{2n+1}(x)}{\sin \pi x}dx,$$

$$\int_0^\infty \tan^{-1}\left(\frac{\tan \pi x}{\tanh \pi t}\right)\frac{t^{2n-1}dt}{\cosh \pi t} = (-1)^n \pi \int_0^x \frac{A'_{2n}(x)}{\cos \pi x}dx,$$

$$\int_0^\infty \tan^{-1}\left(\frac{\tan \pi x}{\tanh \pi t}\right)\frac{t^{2n}dt}{\sinh 2\pi t} = (-1)^{n+1} \pi \int_0^x \frac{A_{2n+1}(x)}{\sin 2\pi x}dx.$$

In these formulæ the limits of x are 0 and 1.

§ 107. We obtain verifications of these formulæ by putting $x = \frac{1}{2}$: they then become

$$\int_0^\infty \frac{t^{2n}dt}{\sinh \pi t} = (-1)^n\, 2\int_0^{\frac{1}{2}} \frac{A'_{2n+1}(x)}{\sin \pi x}dx,$$

$$\int_0^\infty \frac{t^{2n-1}dt}{\cosh \pi t} = (-1)^n\, 2\int_0^{\frac{1}{2}} \frac{A'_{2n}(x)}{\cos \pi x}dx,$$

$$\int_0 \frac{t^{2n}dt}{\sinh 2\pi t} = (-1)^{n+1} \int_0^{\frac{1}{2}} \frac{A_{2n+1}(x)}{\sin 2\pi x}dx,$$

and by § 95 the right-hand members of these equations are equal to

$$2(2n)!\,\frac{v_{2n+1}}{\pi^{2n+1}},\quad 2(2n-1)!\,\frac{u_{2n}}{\pi^{2n}},\quad 2(2n)!\,\frac{v_{2n+1}}{(2\pi)^{2n+1}},$$

respectively. We thus have

$$\int_0^\infty \frac{t^{2n-1}dt}{e^{\pi t}+e^{-\pi t}} = (2n-1)!\,\frac{u_{2n}}{\pi^{2n}},$$

$$\int_0^\infty \frac{t^{2n}dt}{e^{\pi t}-e^{-\pi t}} = (2n)!\,\frac{v_{2n+1}}{\pi^{2n+1}},$$

$$\int_0^\infty \frac{t^{2n}dt}{e^{2\pi t}-e^{-2\pi t}} = (2n)!\,\frac{v_{2n+1}}{(2\pi)^{2n+1}},$$

which are at once seen to be true by expanding the hyperbolic functions under the sign of integration in ascending powers of $e^{-\pi t}$ and performing the integrations.

§ 108. Similarly, since

$$\pi\int_0^x \frac{dx}{\cosh 2\pi t+\cos 2\pi x} = \frac{1}{\sinh 2\pi t}\tan^{-1}(\tanh \pi t\tan x),$$

we find, by putting $x+\frac{1}{2}$ for x in the formulæ quoted in § 106 and integrating between the limits 0 and x,

$$\int_0^\infty \tan^{-1}(\tanh \pi t\tan \pi x)\,\frac{t^{2n}dt}{\sinh \pi t} = (-1)^n\,\pi\int_0^x \frac{\alpha'_{2n+1}(x)}{\sin \pi x}\,dx,$$

$$\int_0^\infty \tan^{-1}(\tanh \pi t\tan \pi x)\,\frac{t^{2n-1}dt}{\cosh \pi t} = (-1)^n\,\pi\int_0^x \frac{\alpha'_{2n}(x)}{\cos \pi x}\,dx,$$

$$\int_0^\infty \tan^{-1}(\tanh \pi t\tan \pi x)\,\frac{t^{2n}dt}{\sinh 2\pi t} = (-1)^{n+1}\,\pi\int_0^x \frac{\alpha_{2n+1}(x)}{\sin 2\pi x}\,dx.$$

In these results the limits of x are $\pm\frac{1}{2}$.

§ 109. We also find, by integration from § 13,

$$\int_0^\infty \tan^{-1}(\tanh \pi t\tan \pi x)\,\frac{\cos at}{\sinh \pi t}\,dt = \frac{\pi}{2\cosh\frac{1}{2}a}\int_0^x \frac{\cosh ax}{\cos \pi x}\,dx,$$

$$\int_0^\infty \tan^{-1}(\tanh \pi t\tan \pi x)\,\frac{\sin at}{\cosh \pi t}\,dt = \frac{\pi}{2\cosh\frac{1}{2}a}\int_0^x \frac{\sinh ax}{\sin \pi x}\,dx,$$

$$\int_0^\infty \tan^{-1}(\tanh \pi t\tan \pi x)\,\frac{\cos at}{\sinh 2\pi t}\,dt = \frac{\pi}{2\sinh\frac{1}{2}a}\int_0^x \frac{\sinh ax}{\sin 2\pi x}\,dx,$$

the limits of x being $\pm\frac{1}{2}$ as in the preceding section.

Expansions of $\dfrac{\sin\frac{p\pi}{q}}{\cosh\frac{a\pi}{q}-\cos\frac{p\pi}{q}}$, &c., *in ascending powers of* x, §§ 110–119.

§ 110. Since, in general, for all values of x and α,

$$\sin\alpha + x\sin 2\alpha + x^2\sin 3\alpha + \ldots + x^{n-1}\sin n\alpha$$
$$= \frac{\sin\alpha - x^n\sin(n+1)\alpha + x^{n+1}\sin n\alpha}{1-2x\cos\alpha + x^2},$$

we find, by putting $\alpha = \dfrac{p\pi}{q}$, where p and q are positive integers, and taking $n = q$,

$$\sin\frac{p\pi}{q} + x\sin\frac{2p\pi}{q} + x^2\sin\frac{3p\pi}{q} + \ldots + x^{q-1}\sin\frac{(q-1)p\pi}{q}$$
$$= \frac{\{1+(-1)^{p-1}x^q\}\sin\frac{p\pi}{q}}{1-2x\cos\frac{p\pi}{q}+x^2}.$$

§ 111. Let

$$x = e^{\frac{a\pi}{q}},$$

and first suppose p uneven. We then have

$$\frac{\sin\frac{p\pi}{q}}{2\left(\cosh\frac{a\pi}{q}-\cos\frac{p\pi}{q}\right)} = \frac{e^{\frac{a\pi}{q}}}{e^{a\pi}+1}\sin\frac{p\pi}{q} + \frac{e^{\frac{2a\pi}{q}}}{e^{a\pi}+1}\sin\frac{2p\pi}{q} + \ldots$$
$$+ \frac{e^{\frac{(q-1)a\pi}{q}}}{e^{a\pi}+1}\sin\frac{(q-1)p\pi}{q}.$$

On the right-hand side, the coefficient of $\dfrac{a^{n-1}\pi^{n-1}}{(n-1)!}$ is (by § 2)

$$A_n'\left(\frac{1}{q}\right)\sin\frac{p\pi}{q} + A_n'\left(\frac{2}{q}\right)\sin\frac{2p\pi}{q} + \ldots + A_n'\left(\frac{q-1}{q}\right)\sin\frac{(q-1)p\pi}{q},$$

and we therefore find, if p be uneven,

$$\frac{\sin\frac{p\pi}{q}}{\cosh\frac{a\pi}{q}-\cos\frac{p\pi}{q}}=2\left\{\lambda_1'+\lambda_3'\frac{a^2\pi^2}{2!}+\lambda_5'\frac{a^4\pi^4}{4!}+\&c.\right\},$$

where

$$\lambda_n'=A_n'\left(\frac{1}{q}\right)\sin\frac{p\pi}{q}+A_n'\left(\frac{2}{q}\right)\sin\frac{2p\pi}{q}+\ldots$$
$$+A_n'\left(\frac{q-1}{q}\right)\sin\frac{(q-1)p\pi}{q}.$$

§ 112. In the case of p even, we have

$$-\frac{\sin\frac{p\pi}{q}}{2\left(\cosh\frac{a\pi}{q}-\cos\frac{p\pi}{q}\right)}=\frac{e^{\frac{a\pi}{q}}}{e^{a\pi}-1}\sin\frac{p\pi}{q}+\frac{e^{\frac{2a\pi}{q}}}{e^{a\pi}-1}\sin\frac{2p\pi}{q}+\ldots$$
$$+\frac{e^{\frac{(q-1)a\pi}{q}}}{e^{a\pi}-1}\sin\frac{(q-1)p\pi}{q};$$

whence, proceeding as before, we find (by § 8)

$$\frac{\sin\frac{p\pi}{q}}{\cosh\frac{a\pi}{q}-\cos\frac{p\pi}{q}}=-2\left\{\lambda_1+\lambda_3\frac{a^2\pi^2}{2!}+\lambda_5\frac{a^4\pi^4}{4!}+\&c.\right\},$$

where

$$\lambda_n=A_n\left(\frac{1}{q}\right)\sin\frac{p\pi}{q}+A_n\left(\frac{2}{q}\right)\sin\frac{2p\pi}{q}+\ldots+A_n\left(\frac{q-1}{q}\right)\sin\frac{(q-1)p\pi}{q}.$$

§ 113. Similarly, from the formula

$$\cos\alpha+x\cos2\alpha+x^2\cos3\alpha+\ldots+x^{n-1}\cos n\alpha$$
$$=\frac{\cos\alpha-x-x^n\cos(n+1)\alpha+x^{n+1}\cos n\alpha}{1-2x\cos\alpha+x^2},$$

where x and a are unrestricted, we find

$$\cos\frac{p\pi}{q}+x\cos\frac{2p\pi}{q}+x^2\cos\frac{3p\pi}{q}+\ldots+x^{q-1}\cos\frac{2p\pi}{q}$$
$$=\frac{\{1+(-1)^{p-1}x^q\}\left(\cos\frac{p\pi}{q}-x\right)}{1-2x\cos\frac{p\pi}{q}+x^2}.$$

Putting $x=e^{\frac{a\pi}{q}}$ and supposing p uneven, we find, by dividing throughout by $e^{a\pi}+1$ and adding $\frac{1}{2}$ to each side of the equation,

$$-\frac{\sinh\frac{a\pi}{q}}{2\left(\cosh\frac{a\pi}{q}-\cos\frac{p\pi}{q}\right)}=\tfrac{1}{2}+\frac{e^{\frac{a\pi}{q}}}{e^{a\pi}+1}\cos\frac{p\pi}{q}+\frac{e^{\frac{2a\pi}{q}}}{e^{a\pi}+1}\cos\frac{2p\pi}{q}+\ldots$$
$$+\frac{e^{\frac{qa\pi}{q}}}{e^{a\pi}+1}\cos\frac{qp\pi}{q}.$$

On the right-hand side, the coefficient of $\dfrac{a^{n-1}\pi^{n-1}}{(n-1)!}$ is

$$A_n'\left(\frac{1}{q}\right)\cos\frac{p\pi}{q}+A_n'\left(\frac{1}{q}\right)\cos\frac{2p\pi}{q}+\ldots+A_n'\left(\frac{2}{q}\right)\cos\frac{qp\pi}{q},$$

so that, if p be uneven,

$$\frac{\sinh\frac{a\pi}{q}}{\cosh\frac{a\pi}{q}-\cos\frac{p\pi}{q}}=-2\left\{\mu_2'a\pi+\mu_4'\frac{a^3\pi^3}{3!}+\mu_6'\frac{a^5\pi^5}{5!}+\&c.\right\},$$

where

$$\mu_n'=A_n'\left(\frac{1}{q}\right)\cos\frac{p\pi}{q}+A_n'\left(\frac{2}{q}\right)\cos\frac{2p\pi}{q}+\ldots+A_n'\left(\frac{q}{q}\right)\cos\frac{qp\pi}{q}.$$

§ 114. When p is even, we have

$$\frac{\sinh\frac{a\pi}{q}}{2\left(\cosh\frac{a\pi}{q}-\cos\frac{p\pi}{q}\right)}=-\tfrac{1}{2}+\frac{e^{\frac{a\pi}{q}}}{e^{a\pi}-1}\cos\frac{p\pi}{q}+\frac{e^{\frac{2a\pi}{q}}}{e^{a\pi}-1}\cos\frac{2p\pi}{q}+\ldots$$
$$+\frac{e^{\frac{qa\pi}{q}}}{e^{a\pi}-1}\cos\frac{qp\pi}{q};$$

whence, proceeding as before, we find

$$\frac{\sinh\frac{a\pi}{q}}{\cosh\frac{a\pi}{q}-\cos\frac{p\pi}{q}}=2\left\{\mu_2 a\pi+\mu_4\frac{a^3\pi^3}{3\,!}+\mu_6\frac{a^5\pi^5}{5\,!}+\&c.\right\},$$

where

$$\mu_n=A_n\left(\frac{1}{q}\right)\cos\frac{p\pi}{q}+A_n\left(\frac{2}{q}\right)\cos\frac{2p\pi}{q}+\ldots+A_n\left(\frac{q}{q}\right)\cos\frac{qp\pi}{q}.$$

§ 115. Multiplying the second formula in § 110 by $1+x$, we have

$$\sin\frac{p\pi}{2q}+x\sin\frac{3p\pi}{2q}+x^2\sin\frac{5p\pi}{2q}+\ldots+x^{n-1}\sin\frac{(2n-1)p\pi}{2q}$$
$$=\frac{(1+x)\{1+(-1)^{p-1}x^n\}\sin\frac{p\pi}{2q}}{1-2x\cos\frac{p\pi}{q}+x^2};$$

whence, multiplying by x^k and proceeding as before, we find, if p be uneven,

$$\frac{\cosh\frac{a\pi}{2q}\sin\frac{p\pi}{2q}}{\cosh\frac{a\pi}{q}-\cos\frac{p\pi}{q}}=\frac{e^{\frac{a\pi}{2q}}}{e^{a\pi}+1}\sin\frac{p\pi}{2q}+\frac{e^{\frac{3a\pi}{2q}}}{e^{a\pi}+1}\sin\frac{3p\pi}{2q}+\ldots$$
$$+\frac{e^{\frac{(2q-1)a\pi}{2q}}}{e^{a\pi}+1}\sin\frac{(2q-1)p\pi}{2q},$$

so that, in the case of p uneven,

$$\frac{\cosh\frac{a\pi}{2q}\sin\frac{p\pi}{2q}}{\cosh\frac{a\pi}{q}-\cos\frac{p\pi}{q}}=\xi_1'+\xi_3'\frac{a^2\pi^2}{2\,!}+\xi_5'\frac{a^4\pi^4}{4\,!}+\&c.,$$

where

$$\xi_n'=A_n'\left(\frac{1}{2q}\right)\sin\frac{p\pi}{2q}+A_n'\left(\frac{3}{2q}\right)\sin\frac{3p\pi}{2q}+\ldots$$
$$+A_n'\left(\frac{2q-1}{2q}\right)\sin\frac{(2q-1)p\pi}{2q}.$$

§ 116. If p be even, the corresponding expression is

$$\frac{\cosh\frac{a\pi}{2q}\sin\frac{p\pi}{2q}}{\cosh\frac{a\pi}{q}-\cos\frac{p\pi}{q}}=-\left\{\xi_1+\xi_3\frac{a^2\pi^2}{2!}+\xi_5\frac{a^4\pi^4}{4!}+\&c.\right\},$$

where

$$\xi_n=A_n\left(\frac{1}{2q}\right)\sin\frac{p\pi}{2q}+A_n\left(\frac{3}{2q}\right)\sin\frac{3p\pi}{2q}+\ldots$$
$$+A_n'\left(\frac{2q-1}{2q}\right)\sin\frac{(2q-1)p\pi}{2q}.$$

§ 117. Similarly, by multiplying the second formula of § 110 by $1-x$, we have

$$\cos\frac{p\pi}{2q}+x\cos\frac{3p\pi}{2q}+x^2\cos\frac{5p\pi}{2q}+\ldots+x^{n-1}\cos\frac{(2n-1)p\pi}{2q}$$
$$=\frac{(1-x)\{1+(-1)^{p-1}x^n\}\cos\frac{p\pi}{2q}}{1-2x\cos\frac{p\pi}{q}+x^2};$$

whence we find

$$\frac{\sinh\frac{a\pi}{2q}\cos\frac{p\pi}{2q}}{\cosh\frac{a\pi}{q}-\cos\frac{p\pi}{q}}=-\left\{\zeta_2'a\pi+\zeta_4'\frac{a^3\pi^3}{3!}+\zeta_6'\frac{a^5\pi^5}{5!}+\&c.\right\}$$

or $$=\zeta_2a\pi+\zeta_4\frac{a^3\pi^3}{3!}+\zeta_6\frac{a^5\pi^5}{5!}+\&c.,$$

according as p is uneven or even, where

$$\zeta_n'=A_n'\left(\frac{1}{2q}\right)\cos\frac{p\pi}{2q}+A_n'\left(\frac{3}{2q}\right)\cos\frac{3p\pi}{2q}+\ldots$$
$$+A_n'\left(\frac{2q-1}{2q}\right)\cos\frac{(2q-1)p\pi}{2q},$$

and

$$\zeta_n=A_n\left(\frac{1}{2q}\right)\cos\frac{p\pi}{2q}+A_n\left(\frac{3}{2q}\right)\cos\frac{3p\pi}{2q}+\ldots$$
$$+A_n\left(\frac{2q-1}{2q}\right)\cos\frac{(2q-1)p\pi}{2q}.$$

§ 118. The expansions obtained in the preceding eight sections may be summarised as follows.

Let x be arbitrary, and $\omega = \frac{p\pi}{q}$, p and q being positive integers: then

$$\frac{\sin \omega}{\cosh x - \cos \omega} = -2\Sigma_0^\infty \lambda_{2n+1} \frac{(qx)^{2n}}{(2n)!} \text{ or } 2\Sigma_0^\infty \lambda'_{2n+1} \frac{(qx)^{2n}}{(2n)!},$$

$$\frac{\sinh x}{\cosh x - \cos \omega} = 2\Sigma_1^\infty \mu_{2n} \frac{(qx)^{2n-1}}{(2n-1)!} \text{ or } -2\Sigma_1^\infty \mu'_{2n} \frac{(qx)^{2n-1}}{(2n-1)!},$$

$$\frac{\cosh \frac{1}{2}x \sin \frac{1}{2}\omega}{\cosh x - \cos \omega} = -\Sigma_0^\infty \xi_{2n+1} \frac{(qx)^{2n}}{(2n)!} \text{ or } \Sigma_0^\infty \xi'_{2n+1} \frac{(qx)^{2n}}{(2n)!},$$

$$\frac{\sinh \frac{1}{2}x \cos \frac{1}{2}\omega}{\cosh x - \cos \omega} = \Sigma_1^\infty \zeta_{2n} \frac{(qx)^{2n-1}}{(2n-1)!} \text{ or } -\Sigma_1^\infty \zeta'_{2n} \frac{(qx)^{2n-1}}{(2n-1)!},$$

the first or second of the expansions being taken according as p is even or uneven.

The values of λ_n, μ_n, ..., are

$$\lambda_n = \Sigma_{r=1}^{r=q} A_n\left(\frac{r}{q}\right) \sin r\omega, \qquad (\S\ 112),$$

$$\mu_n = \Sigma_{r=1}^{r=q} A_n\left(\frac{r}{q}\right) \cos r\omega, \qquad (\S\ 114),$$

$$\xi_n = \Sigma_{r=1}^{r=q} A_n\left(\frac{2r-1}{2q}\right) \sin \frac{(2r-1)\,\omega}{2}, \quad (\S\ 116),$$

$$\zeta_n = \Sigma_{r=1}^{r=q} A_n\left(\frac{2r-1}{2q}\right) \cos \frac{(2r-1)\,\omega}{2}, \quad (\S\ 117).$$

The values of λ_n', μ_n', ... differ from those of λ_n, μ_n, ... only by the substitution of the function A_n' for A_n.

§ 119. By integrating with respect to x the second formula in the preceding section, we find

$$\log(\cosh x - \cos \omega) = \log(1 - \cos \omega) + 2\Sigma_1^\infty \mu_{2n} \frac{(qx)^{2n}}{(2n)!}$$

$$\text{or } \log(1 - \cos \omega) - 2\Sigma_1^\infty \mu'_{2n} \frac{(qx)^{2n}}{(2n)!}$$

according as p is even or uneven.

Expansions of $\dfrac{\sin\dfrac{p\pi}{q}}{\cosh\dfrac{a\pi}{q}+\cos\dfrac{p\pi}{q}}$, &c. *in ascending powers of* x,

§§ 120–121.

§ 120. By substituting $q-p$ for p in the formulæ of § 118, we obtain the following expansions in which the form of the coefficients depends upon whether p and q are both even or uneven, or one is even and the other uneven, instead of, as in § 118, upon whether p is even or uneven.

Let x be arbitrary and $\omega=\dfrac{p\pi}{q}$, p and q being positive integers: then

$$\frac{\sin\omega}{\cosh x+\cos\omega}=-2\Sigma_0^\infty\,[\lambda_{2n+1}]\frac{(qx)^{2n}}{(2n)!}\text{ or }2\Sigma_0^\infty\,[\lambda'_{2n+1}]\frac{(qx)^{2n}}{(2n)!},$$

$$\frac{\sinh x}{\cosh x+\cos\omega}=-2\Sigma_1^\infty\,[\mu_{2n}]\frac{(qx)^{2n-1}}{(2n-1)!}\text{ or }2\Sigma_1^\infty\,[\mu'_{2n}]\frac{(qx)^{2n-1}}{(2n-1)!},$$

$$\frac{\cosh\frac{1}{2}x\cos\frac{1}{2}\omega}{\cosh x+\cos\omega}=-\Sigma_0^\infty\,[\xi_{2n+1}]\frac{(qx)^{2n}}{(2n)!}\text{ or }\Sigma_0^\infty\,[\xi'_{2n+1}]\frac{(qx)^{2n}}{(2n)!},$$

$$\frac{\sinh\frac{1}{2}x\sin\frac{1}{2}\omega}{\cosh x+\cos\omega}=\Sigma_1^\infty\,[\zeta_{2n}]\frac{(qx)^{2n-1}}{(2n-1)!}\text{ or }-\Sigma_1^\infty\,[\zeta'_{2n}]\frac{(qx)^{2n-1}}{(2n-1)!},$$

the first or second of the expansions being taken according as p and q are both even or uneven, or one is even and the other uneven.

The values of $[\lambda_n]$, $[\mu_n]$, ..., are

$$[\lambda_n]=\Sigma_{r=1}^{r=q}(-1)^{r-1}A_n\left(\frac{r}{q}\right)\sin r\omega,$$

$$[\mu_n]=\Sigma_{r=1}^{r=q}(-1)^{r-1}A_n\left(\frac{r}{q}\right)\cos r\omega,$$

$$[\xi_n]=\Sigma_{r=1}^{r=q}(-1)^{r-1}A_n\left(\frac{2r-1}{2q}\right)\sin\frac{(2r-1)\,\omega}{2},$$

$$[\zeta_n]=\Sigma_{r=1}^{r=q}(-1)^{r-1}A_n\left(\frac{2r-1}{2q}\right)\cos\frac{(2r-1)\,\omega}{2}.$$

The values of $[\lambda_n']$, $[\mu_n']$, ... differ from those of $[\lambda_n]$, $[\mu_n]$, ... only by the substitution of the function A_n' for A_n.

The terms in the coefficients $[\lambda_n]$, ... are the same as in λ_n, ..., but the signs are alternately positive and negative.

§ 121. From the second formula, we find, as in § 119

$$\log(\cosh x + \cos\omega) = \log(1+\cos\omega) - 2\Sigma_0^\infty\, [\mu_{2n}]\frac{(qx)^{2n}}{(2n)!},$$

$$\text{or } \log(1+\cos\omega) + 2\Sigma_1^\infty\, [\mu'_{2n}]\frac{(qx)^{2n}}{(2n)!}$$

according as p and q are both even or uneven, or one is even and the other uneven.

Evaluation of the integrals $\int_0^\infty \frac{\sinh(q-p)\,t}{\sinh qt}\, t^{2n} dt$, &c., §§ 122–129.

§ 122. From § 14, we find, by putting $q-p$ for p,

$$\text{(i)}\quad \int_0^\infty \frac{\sinh(q-p)\,t}{\sinh qt}\cos at\,dt = \frac{\pi}{2q}\,\frac{\sin\dfrac{p\pi}{q}}{\cosh\dfrac{a\pi}{q}-\cos\dfrac{p\pi}{q}},$$

$$\text{(ii)}\quad \int_0^\infty \frac{\cosh(q-p)\,t}{\sinh qt}\sin at\,dt = \frac{\pi}{2q}\,\frac{\sinh\dfrac{a\pi}{q}}{\cosh\dfrac{a\pi}{q}-\cos\dfrac{p\pi}{q}},$$

$$\text{(iii)}\quad \int_0^\infty \frac{\cosh(q-p)\,t}{\cosh qt}\cos at\,dt = \frac{\pi}{q}\,\frac{\cosh\dfrac{a\pi}{2q}\sin\dfrac{p\pi}{2q}}{\cosh\dfrac{a\pi}{q}-\cos\dfrac{p\pi}{q}},$$

$$\text{(iv)}\quad \int_0^\infty \frac{\sinh(q-p)\,t}{\cosh qt}\sin at\,dt = \frac{\pi}{q}\,\frac{\sinh\dfrac{a\pi}{2q}\cos\dfrac{p\pi}{2q}}{\cosh\dfrac{a\pi}{q}-\cos\dfrac{p\pi}{q}},$$

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In these formulæ p and q may be any real quantities subject to the condition that $q-p$ must be numerically less than q.

§ 123. Expanding $\cos at$ in powers of a, the first integral becomes

$$\int_0^\infty \frac{\sinh (q-p)\,t}{\sinh qt}\left(1-\frac{a^2t^2}{2!}+\frac{a^4t^4}{4!}-\&c.\right)dt;$$

and, supposing p and q to be integers, the expansion of the right-hand side of (i) is, by §§ 111 and 112,

$$\frac{\pi}{q}\,\Sigma_0^\infty \lambda'_{2n+1}\frac{a^{2n}\pi^{2n}}{(2n)!} \text{ or } -\frac{\pi}{q}\,\Sigma_0^\infty \lambda_{2n+1}\frac{a^{2n}\pi^{2n}}{(2n)!},$$

according as p is even or uneven.

Thus we find

$$\int_0^\infty t^{2n}\frac{\sinh (q-p)\,t}{\sinh qt}\,dt=(-1)^n\frac{\pi^{2n+1}}{q}\,\lambda'_{2n+1} \text{ or } (-1)^{n-1}\frac{\pi^{2n+1}}{q}\,\lambda_{2n+1},$$

according as p is uneven or even.

As an example, putting $q=3$, $p=1$,

$$\int_0^\infty t^{2n}\frac{\sinh 2t}{\sinh 3t}\,dt=(-1)^n\frac{\pi^{2n+1}}{3}\left\{A'_{2n+1}(\tfrac{1}{3})\sin\frac{\pi}{3}+A'_{2n+1}(\tfrac{2}{3})\sin\frac{2\pi}{3}\right\}$$

$$=(-1)^n\frac{\pi^{2n+1}}{\sqrt{3}}A'_{2n+1}(\tfrac{1}{3})=\frac{\pi^{2n+1}H_n}{3^{2n+1}\sqrt{3}}, \quad (\S\,55),$$

agreeing with § 45 (p. 175).

As a second example, putting $q=4$, $p=1$, the formula gives

$$\int_0^\infty t^{2n}\frac{\sinh 3t}{\sinh 4t}\,dt=(-1)^n\frac{\pi^{2n+1}}{4}$$

$$\times\left\{A'_{2n+1}(\tfrac{1}{4})\sin\frac{\pi}{4}+A'_{2n+1}(\tfrac{1}{2})\sin\frac{\pi}{2}+A'_{2n+1}(\tfrac{3}{4})\sin\frac{3\pi}{4}\right\}$$

$$=(-1)^n\frac{\pi^{2n+1}}{4}\left\{\sqrt{2}\,.\,A'_{2n+1}(\tfrac{1}{4})+A'_{2n+1}(\tfrac{1}{2})\right\}$$

$$=\pi^{2n+1}\left(\frac{P_n}{4^{2n+1}\sqrt{2}}+\frac{E_n}{2^{2n+3}}\right), \quad (\S\,55).$$

This result may be readily verified; for

$$\frac{\pi^{2n+1}P_n\sqrt{2}}{4^{2n+1}} = \int_0^\infty t^{2n}\frac{\cosh t}{\cosh 2t}\,dt, \quad (\S\,46)$$

$$= \int_0^\infty t^{2n}\frac{\sinh 3t + \sinh t}{\sinh 4t}\,dt,$$

and

$$\frac{\pi^{2n+1}E_n}{2^{2n+2}} = \tfrac{1}{2}\int_0^\infty \frac{t^{2n}dt}{\cosh t}, \quad (\S\,42)$$

$$= \int_0^\infty t^{2n}\frac{\sinh 3t - \sinh t}{\sinh 4t}\,dt,$$

from which the result in question may be at once obtained by addition.

Similarly, by subtraction, we obtain the result

$$\int_0^\infty t^{2n}\frac{\sinh t}{\sinh 4t}\,dt = \pi^{2n+1}\left(\frac{P_n}{4^{2n+1}\sqrt{2}} - \frac{E_n}{2^{2n+3}}\right),$$

which is derivable from the general formula by putting $q = 4$, $p = 3$.

§ 124. As an example of the formula in the case of p even, let $q = 3$, $p = 2$. We then have

$$\int_0^\infty t^{2n}\frac{\sinh t}{\sinh 3t}\,dt = (-1)^{n-1}\frac{\pi^{2n+1}}{3}\left\{A_{2n+1}(\tfrac{1}{3})\sin\frac{2\pi}{3} + A_{2n+1}(\tfrac{2}{3})\sin\frac{4\pi}{3}\right\}$$

$$= (-1)^{n-1}\frac{\pi^{2n+1}}{\sqrt{3}}A_{2n+1}(\tfrac{1}{3}) = \frac{\pi^{2n+1}I_n}{3^{2n+1}\sqrt{3}}, \quad (\S\,55)$$

agreeing with § 44 (p. 174).

As a second example, let $q = 6$, $p = 2$; the formula then gives

$$\int_0^\infty t^{2n}\frac{\sinh 4t}{\sinh 6t}\,dt = (-1)^{n-1}\frac{\pi^{2n+1}}{6}$$

$$\times\left\{A_{2n+1}(\tfrac{1}{6})\sin\frac{2\pi}{6} + A_{2n+1}(\tfrac{2}{6})\sin\frac{4\pi}{6} + \ldots + A_{2n+1}(\tfrac{5}{6})\sin\frac{10\pi}{6}\right\}$$

$$= (-1)^{n-1}\frac{\pi^{2n+1}}{6}\sqrt{3}\,\{A_{2n+1}(\tfrac{1}{6}) + A_{2n+1}(\tfrac{1}{3})\}$$

$$= \frac{\pi^{2n+1}}{2\sqrt{3}}\left(\frac{J_n}{6^{2n+1}} + \frac{I_n}{3^{2n+1}}\right), \quad (\S\,55),$$

which can be readily identified with the result

$$\int_0^\infty t^{2n} \frac{\sinh 2t}{\sinh 3t} dt = \frac{\pi^{2n+1} H_n}{3^{2n+1}\sqrt{3}}, \quad (\S\ 45),$$

since $J_n + 2^{2n+1} I_n = 2H_n$, (§ 30).

§ 125. Proceeding as in § 123, we deduce from the second formula of § 122 that

$$\int_0^\infty t^{2n-1} \frac{\cosh (q-p)\, t}{\sinh qt} dt = (-1)^n \frac{\pi^{2n}}{q} \mu'_{2n} \text{ or } (-1)^{n-1} \frac{\pi^{2n}}{q} \mu_{2n},$$

according as p is uneven or even.

For example, putting $q=3$, $p=1$,

$$\int_0^\infty t^{2n-1} \frac{\cosh 2t}{\sinh 3t} dt = (-1)^n \frac{\pi^{2n}}{3}$$

$$\times \left\{A'_{2n}(\tfrac{1}{3}) \cos\frac{\pi}{3} + A'_{2n}(\tfrac{2}{3}) \cos\frac{2\pi}{3} + A'_{2n}(\tfrac{3}{3}) \cos\frac{3\pi}{3}\right\}$$

$$= (-1)^n \frac{\pi^{2n}}{3} \{\tfrac{1}{2}A'_{2n}(\tfrac{1}{3}) - \tfrac{1}{2}A'_{2n}(\tfrac{2}{3}) - A'_{2n}(1)\}$$

$$= (-1)^n \frac{\pi^{2n}}{3} \{A'_{2n}(0) + A'_{2n}(\tfrac{1}{3})\}$$

$$= \frac{\pi^{2n}}{3} \left\{(2^{2n}-1)\frac{B_n}{2n} + \frac{(2^{2n}-1)(3^{2n}-3)}{3^{2n}} \frac{B_n}{4n}\right\}$$

$$= \frac{\pi^{2n}(2^{2n}-1)(3^{2n}-1)}{3^{2n}} \frac{B_n}{4n}.$$

By putting $q=3$, $p=2$, the formula gives

$$\int_0^\infty t^{2n-1} \frac{\cosh t}{\sinh 3t} dt = (-1)^{n-1} \frac{\pi^{2n}}{3} \{-\tfrac{1}{2}A_{2n}(\tfrac{1}{3}) - \tfrac{1}{2}A_{2n}(\tfrac{2}{3}) + A_{2n}(1)\}$$

$$= (-1)^{n-1} \frac{\pi^{2n}}{3} \{A_{2n}(0) - A_{2n}(\tfrac{1}{3})\}^* = (-1)^{n-1} \frac{\pi^{2n}(3^{2n}-1)}{3^{2n}} \frac{B_n}{4n}.$$

* In the list of § 55 (vol. xxvi., p. 179 l. 4), the value of $3^{2n}A_{2n}(\tfrac{1}{3})$ has been quoted as $(-1)^n(3^{2n}-3)\dfrac{B_n}{2n}$, instead of $(-1)^n(3^{2n}-3)\dfrac{B_n}{4n}$.

§ 126. From the third formula of § 122 combined with §§ 115 and 116, we find

$$\int_0^\infty t^{2n} \frac{\cosh (q-p)\, t}{\cosh qt}\, dt = (-1)^n \frac{\pi^{2n+1}}{q} \xi'_{2n+1} \text{ or } (-1)^{n-1} \frac{\pi^{2n+1}}{q} \xi_{2n+1},$$

according as p is uneven or even.

For example, putting $q = 3$, $p = 1$,

$$\int_0^\infty t^{2n} \frac{\cosh 2t}{\cosh 3t}\, dt = (-1)^n \frac{\pi^{2n+1}}{3}$$

$$\times \left\{A'_{2n+1}(\tfrac{1}{6}) \sin \frac{\pi}{6} + A'_{2n+1}(\tfrac{1}{2}) \sin \frac{3\pi}{6} + A'_{2n+1}(\tfrac{5}{6}) \sin \frac{5\pi}{6}\right\}$$

$$= (-1)^n \frac{\pi^{2n+1}}{3} \{A'_{2n+1}(\tfrac{1}{6}) + A'_{2n+1}(\tfrac{1}{2})\}$$

$$= \frac{\pi^{2n+1}}{3} \left\{\frac{3^{2n+1}+3}{2} \frac{E_n}{6^{2n+1}} + \frac{E_n}{2^{2n+1}}\right\} = \frac{\pi^{2n+1}(3^{2n+1}+1)\, E_n}{2 \times 6^{2n+1}},$$

agreeing with (iii) of § 38 (p. 171), since $R_n = \frac{1}{2}(3^{2n+1}+1)\, E_n$, (§ 32).

§ 127. Similarly, from (iv) of § 122 combined with § 117, we find

$$\int_0^\infty t^{2n-1} \frac{\sinh (q-p)\, t}{\cosh qt}\, dt = (-1)^n \frac{\pi^{2n}}{q} \zeta'_{2n} \text{ or } (-1)^{n-1} \frac{\pi^{2n}}{q} \zeta_{2n},$$

according as p is uneven or even.

For example, putting $q = 3$, $p = 1$,

$$\int_0^\infty t^{2n-1} \frac{\sinh 2t}{\cosh 3t}\, dt = (-1)^n \frac{\pi^{2n}}{3}$$

$$\times \left\{A'_{2n}(\tfrac{1}{6}) \cos \frac{\pi}{6} + A'_{2n}(\tfrac{1}{2}) \cos \frac{3\pi}{6} + A'_{2n}(\tfrac{5}{6}) \cos \frac{5\pi}{6}\right\}$$

$$= (-1)^n \frac{\pi^{2n}}{\sqrt{3}} A'_{2n}(\tfrac{1}{6}) = \frac{\pi^{2n} T_{2n}}{6^{2n-1}\sqrt{3}},$$

agreeing with (iv) of § 38 (p. 171).

§ 128. Collecting together the results of the six preceding sections, we obtain the following system of integral-evaluations,

in which the first or second value is to be taken for the integral according as p is uneven or even.

$$\text{(i)}\quad \int_0^\infty t^{2n}\frac{\sinh(q-p)\,t}{\sinh qt}\,dt=(-1)^n\frac{\pi^{2n+1}}{q}\lambda'_{2n+1}$$

$$\text{or}\ (-1)^{n-1}\frac{\pi^{2n+1}}{q}\lambda_{2n+1},\qquad (\S\,123),$$

$$\text{(ii)}\quad \int_0^\infty t^{2n-1}\frac{\cosh(q-p)\,t}{\sinh qt}\,dt=(-1)^n\frac{\pi^{2n}}{q}\mu'_{2n}$$

$$\text{or}\ (-1)^{n-1}\frac{\pi^{2n}}{q}\mu_{2n},\qquad (\S\,125),$$

$$\text{(iii)}\quad \int_0^\infty t^{2n}\frac{\cosh(q-p)\,t}{\cosh qt}\,dt=(-1)^n\frac{\pi^{2n+1}}{q}\xi'_{2n+1}$$

$$\text{or}\ (-1)^{n-1}\frac{\pi^{2n+1}}{q}\xi_{2n+1},\qquad (\S\,126),$$

$$\text{(iv)}\quad \int_0^\infty t^{2n-1}\frac{\sinh(q-p)\,t}{\cosh qt}\,dt=(-1)^n\frac{\pi^{2n}}{q}\zeta'_{2n}$$

$$\text{or}\ (-1)^{n-1}\frac{\pi^{2n}}{q}\zeta_{2n},\qquad (\S\,127).$$

The values of λ_n', λ_n, ... were given in § 118. These quantities are also considered in §§ 131–138.

§ 129. It is evident that by means of these formulæ we may assign the values of the four integrals when the quantities $q-p$ and q, which occur as the coefficients of t in the numerators and denominators respectively, are any commensurable quantities whatever (subject, of course, to the condition that the integrals are finite, *i.e.* that $q-p$ is numerically less than q); for, by putting $t=mt'$ where m is the least common multiple of the denominators of $q-p$ and q, we may transform the integrals into others in which the factors by which t is multiplied in the numerator and denominator are both integers.

Values of the integrals $\int_0^\infty \frac{\sinh pt}{\sinh qt}t^{2n}dt$, &c., § 130.

§ 130. The form in which the integrals were written in § 128 is a convenient one for use; but we may, of course,

render the expression of the integrals more simple by substituting p for $q-p$. The system of formulæ then becomes

$$\text{(i)}\quad \int_0^\infty t^{2n}\,\frac{\sinh pt}{\sinh qt}\,dt=(-1)^{n-1}\frac{\pi^{2n+1}}{q}[\lambda_{2n+1}]\text{ or }(-1)^n\frac{\pi^{2n+1}}{q}[\lambda'_{2n+1}],$$

$$\text{(ii)}\quad \int_0^\infty t^{2n-1}\frac{\cosh pt}{\sinh qt}\,dt=(-1)^n\frac{\pi^{2n}}{q}[\mu_{2n}]\quad\text{or}\quad(-1)^{n-1}\frac{\pi^{2n}}{q}[\mu'_{2n}],$$

$$\text{(iii)}\quad \int_0^\infty t^{2n}\,\frac{\cosh pt}{\cosh qt}\,dt=(-1)^{n-1}\frac{\pi^{2n+1}}{q}[\xi_{2n+1}]\text{ or }(-1)^n\frac{\pi^{2n+1}}{q}[\xi'_{2n+1}],$$

$$\text{(iv)}\quad \int_0^\infty t^{2n-1}\frac{\sinh pt}{\cosh qt}\,dt=(-1)^{n-1}\frac{\pi^{2n}}{q}[\zeta_{2n}]\quad\text{or}\quad(-1)^n\frac{\pi^{2n}}{q}[\zeta'_{2n}],$$

the first or second value being taken according as p and q are both even or uneven, or one even and the other uneven.

The values of $[\lambda_n]$, $[\lambda_n']$, ... were given in § 120.

The functions $\lambda_n(q,p)$, $\mu_n(q,p)$, &c., § 131.

§ 131. In considering the quantities λ_n, μ_n, ..., and the relations between them, we require to put in evidence the arguments q and p, and it is convenient to write $\lambda_n(q,p)$, ..., in place of λ_n, ..., so that

$$\lambda_n(q,p)=A_n\left(\frac{1}{q}\right)\sin\frac{p\pi}{q}+A_n\left(\frac{2}{q}\right)\sin\frac{2p\pi}{q}+\dots$$
$$+A_n\left(\frac{q-1}{q}\right)\sin\frac{(q-1)\,p\pi}{q},$$

$$\mu_n(q,p)=A_n\left(\frac{1}{q}\right)\cos\frac{p\pi}{q}+A_n\left(\frac{2}{q}\right)\cos\frac{2p\pi}{q}+\dots+A_n\left(\frac{q}{q}\right)\cos\frac{qp\pi}{q},$$

$$\xi_n(q,p)=A_n\left(\frac{1}{2q}\right)\sin\frac{p\pi}{2q}+A_n\left(\frac{3}{2q}\right)\sin\frac{3p\pi}{2q}+\dots$$
$$+A_n\left(\frac{2q-1}{2q}\right)\sin\frac{(2q-1)\,p\pi}{2q},$$

$$\zeta_n(q,p)=A_n\left(\frac{1}{2q}\right)\cos\frac{p\pi}{2q}+A_n\left(\frac{3}{2q}\right)\cos\frac{3p\pi}{2q}+\dots$$
$$+A_n\left(\frac{2q-1}{2q}\right)\cos\frac{(2q-1)\,p\pi}{2q},$$

with similar definitions of $\lambda_n(q, p)$, ..., in which A_n is replaced by A_n'.

We suppose n, q, p to be positive integers, but otherwise unrestricted.

Cases in which $\lambda_n(q, p)$, &c. vanish, §§ 132–138.

§ 132. In the expression represented by $\lambda_n(q, p)$ the r^{th} terms from the beginning and end are respectively

$$A_n\left(\frac{r}{p}\right)\sin\frac{rp\pi}{q} \text{ and } A_n\left(\frac{q-r}{q}\right)\sin\frac{(q-r)p\pi}{q}.$$

The latter term

$$=(-1)^n A_n\left(\frac{r}{q}\right).(-1)^{p-1}\sin\frac{rp\pi}{q},$$

so that these two terms cancel each other if $n+p$ is even.

When q is even there is also the middle term $A_n(\frac{1}{2})\sin\frac{1}{2}p\pi$ which vanishes (i) if n is uneven, (ii) if p is even.

The only case therefore in which this term does not vanish is when n is even and p is uneven.

Thus we see that $\lambda_n(q, p)=0$, if $n+p$ is even, that is, if n and p are both even or both uneven.

§ 133. In the expression for $\lambda_n'(q, p)$ the terms equidistant from the beginning and end cancel each other if $n+p$ is uneven, and the middle term vanishes (i) if n is even, (ii) if p is even.

Thus $\lambda_n'(q, p)=0$, if $n+p$ is uneven, that is, if n and p are one even and the other uneven.

§ 134. In the function $\mu_n(q, p)$ the terms

$$A_n\left(\frac{r}{p}\right)\cos\frac{rp\pi}{q} \text{ and } A_n\left(\frac{q-r}{q}\right)\cos\frac{(q-r)p\pi}{q}$$

cancel each other if $n+p$ is uneven.

The middle term $A_n(\frac{1}{2})\cos\frac{1}{2}p\pi$ which occurs when q is even vanishes (i) if n is uneven, (ii) if p is uneven.

The last term $A_n\left(\frac{q}{q}\right)\cos\frac{qp\pi}{q}=A_n(1)\cos p\pi$ vanishes only when n is uneven.

Thus $\mu_n(q, p) = 0$, if n is uneven and p is even.

It will be noticed that, if n is even and p is uneven,

$$\mu_n(q, p) = -A_n(1) = (-1)^{\frac{1}{2}n} \frac{B_{\frac{1}{2}n}}{n}.$$

§ 135. In the case of $\mu_n'(q, p)$, the r^{th} and $(q-r)^{\text{th}}$ terms cancel each other if $n+p$ is even. The middle term vanishes (i) if n is even, (ii) if p is uneven. The last term vanishes when n is uneven.

Thus $\mu_n'(q, p) = 0$, if n is uneven and p is uneven.

We see also that, if n is even and p is even,

$$\mu_n'(q, p) = -A_n'(1) = (-1)^{\frac{1}{2}n} (2^n - 1) \frac{B_{\frac{1}{2}n}}{n}.$$

§ 136. In the function $\xi_n(q, p)$ the r^{th} terms from the beginning and end are respectively

$$A_n\left(\frac{r}{2q}\right) \sin\frac{rp\pi}{2q} \text{ and } A_n\left(\frac{2q-r}{2q}\right) \sin\frac{(2q-r)\,p\pi}{2q},$$

the latter of which

$$= (-1)^n A_n\left(\frac{r}{2q}\right) . (-1)^{p-1} \sin\frac{rp\pi}{q}.$$

These terms therefore cancel each other when $n+p$ is even.

The middle term $A_n(\frac{1}{2}) \sin\frac{1}{2}p\pi$, when it occurs, vanishes (i) if n is uneven, (ii) if p is even.

Thus $\xi_n(q, p) = 0$, if $n+p$ is even, that is, if n and p are both even or both uneven.

In the same manner we see that $\xi'_{n}(q, p) = 0$, if n and p are one even and the other uneven.

§ 137. In the function $\zeta_n(q, p)$ the terms equidistant from the beginning and end cancel each other if $n+p$ is uneven, and the middle term vanishes (i) if n is uneven, (ii) if p is uneven.

Thus $\zeta_n(q, p) = 0$, if n and p are one even and one uneven.

We also see that $\zeta_n'(q, p) = 0$, if n and p are both even or both uneven.

§ 138. We may express these results in a tabular form as follows:

	n even	n uneven
p uneven	$\lambda_n = 0$ $\mu_n' = 0$ $\xi_n = 0$ $\zeta_n' = 0$	$\lambda_n' = 0$ $\mu_n = (-1)^{\frac{1}{2}n} \dfrac{B_{\frac{1}{2}n}}{n}$ $\xi_n' = 0$ $\zeta_n = 0$
p even	$\lambda_n' = 0$ $\mu_n = 0$ $\xi_n' = 0$ $\zeta_n = 0$	$\lambda_n = 0$ $\mu_n' = (-1)^{\frac{1}{2}n} (2^n - 1) \dfrac{B_{\frac{1}{2}n}}{n}$ $\xi_n = 0$ $\zeta_n' = 0$

Relations between the functions $\lambda_n(q, p)$, &c., §§ 139–141.

§ 139. From the definition of $\lambda_n(q, p)$ in § 131, we see that

$$\lambda_n(q, q-p) = A_n\left(\frac{1}{q}\right)\sin\frac{p\pi}{q} - A_n\left(\frac{2}{q}\right)\sin\frac{2p\pi}{q} + \dots,$$

so that
$$\lambda_n(2q, p) - \lambda_n(2q, 2q-p) = 2\lambda_n(q, p),$$
$$\lambda_n(2q, p) + \lambda_n(2q, 2q-p) = 2\xi_n(q, p).$$

Similarly

$$\mu_n(2q, p) + \mu_n(2q, 2q-p) = 2\mu_n(q, p),$$
$$\mu_n(2q, p) - \mu_n(2q, 2q-p) = 2\zeta_n(q, p).$$

§ 140. From the above equations, or directly from the definitions, we have also

$$\lambda_n(2q, p) - \lambda_n(q, p) = \xi_n(q, p),$$
$$\mu_n(2q, p) - \mu_n(q, p) = \zeta_n(q, p).$$

§ 141. The quantities $[\lambda_n]$, &c., of § 120 are obviously connected with λ_n, &c., by the equations

$$\begin{aligned}[\lambda_n(q, p)] &= \lambda_n(q, q-p),\\ [\mu_n(q, p)] &= -\mu_n(q, q-p),\\ [\xi_n(q, p)] &= \xi_n(q, q-p),\\ [\zeta_n(q, p)] &= \zeta_n(q, q-p),\end{aligned}$$

so that the relations of § 139 may be written

$$\lambda_n(2q, p) - [\lambda_n(2q, p)] = 2\lambda_n(q, p), \text{ \&c.}$$

Properties of the functions $\lambda_n(q, p)$, $\mu_n(q, p)$, &c., §§ 142–147.

§ 142. It is evident from the definitions of $\lambda_n(q, p)$, &c., in § 131 that

$$\begin{aligned}\lambda_n(q, p+2q) &= \lambda_n(q, p),\\ \mu_n(q, p+2q) &= \mu_n(q, p),\\ \xi_n(q, p+2q) &= -\xi_n(q, p),\\ \zeta_n(q, p+2q) &= -\zeta_n(q, p).\end{aligned}$$

§ 143. It was shown in §§ 111 and 112 that

$$\frac{\sin\dfrac{p\pi}{q}}{\cosh\dfrac{a\pi}{q} - \cos\dfrac{p\pi}{q}} = 2\Sigma_0^\infty \lambda'_{2n+1}(q, p)\frac{(a\pi)^{2n}}{(2n)!}$$

$$\text{or } -2\Sigma_0^\infty \lambda_{2n+1}(q, p)\frac{(a\pi)^{2n}}{(2n)!},$$

according as p is uneven or even.

In this equation replace q, p, a by $2q$, $2p$, $2a$. The left-hand side remains unaltered, and we therefore have

$$\frac{\sin\dfrac{p\pi}{q}}{\cosh\dfrac{a\pi}{q} - \cos\dfrac{p\pi}{q}} = -2\Sigma_0^\infty \lambda_{2n+1}(2q, 2p)\frac{(2a\pi)^{2n}}{(2n)!},$$

whence we obtain the relation

$$2^{2n}\lambda_{2n+1}(2q, 2p) = \lambda_{2n+1}(q, p) \text{ or } -\lambda'_{2n+1}(q, p),$$

according as p is even or uneven.

Similar relations hold good with respect to $\mu_{2n}(q, p)$, &c., for the generating functions (§§ 114–117) all remain unaltered by the simultaneous change of q, p, a into $2q, 2p, 2a$.

We thus have the system of formulæ

$$2^{2n}\lambda_{2n+1}(2q, 2p) = \lambda_{2n+1}(q, p) \text{ or } -\lambda'_{2n+1}(q, p),$$

$$2^{2n-1}\mu_{2n}(2q, 2p) = \mu_{2n}(q, p) \text{ or } -\mu'_{2n}(q, p),$$

$$2^{2n}\xi_{2n+1}(2q, 2p) = \xi_{2n+1}(q, p) \text{ or } -\xi'_{2n+1}(q, p),$$

$$2^{2n-1}\zeta_{2n}(2q, 2p) = \zeta_{2n}(q, p) \text{ or } -\zeta'_{2n}(q, p),$$

the first or second value being taken according as p is even or uneven.

§ 144. By replacing p, q, a in § 133 by $3p, 3q, 3a$, we see in the same way that, if p be uneven,

$$3^{2n}\lambda'_{2n+1}(3q, 3p) = \lambda'_{2n+1}(q, p);$$

and that, if p be even,

$$3^{2n}\lambda_{2n+1}(3q, 3p) = \lambda_{2n+1}(q, p),$$

and so on.

§ 145. Thus, in general, if r be even

$$r^{2n}\lambda_{2n+1}(rq, rp) = \lambda_{2n+1}(q, p) \text{ or } -\lambda'_{2n+1}(q, p),$$

$$r^{2n-1}\mu_{2n}(rq, rp) = \mu_{2n}(q, p) \text{ or } -\mu'_{2n}(q, p),$$

$$r^{2n}\xi_{2n+1}(rq, rp) = \xi_{2n+1}(q, p) \text{ or } -\xi'_{2n+1}(q, p),$$

$$r^{2n-1}\zeta_{2n}(rq, rp) = \zeta_{2n}(q, p) \text{ or } -\zeta'_{2n}(q, p),$$

according as p is even or uneven.

If r be uneven and p be even,

$$r^{2n}\lambda_{2n+1}(rq, rp) = \lambda_{2n+1}(q, p),$$

$$r^{2n-1}\mu_{2n}(rq, rp) = \mu_{2n}(q, p),$$

$$r^{2n}\xi_{2n+1}(rq, rp) = \xi_{2n+1}(q, p),$$

$$r^{2n-1}\zeta_{2n}(rq, rp) = \zeta_{2n}(q, p);$$

and, if r be uneven and p be uneven,

$$r^{2n}\lambda'_{2n+1}(rq, rp) = \lambda'_{2n+1}(q, p),$$
$$r^{2n-1}\mu'_{2n}(rq, rp) = \mu'_{2n}\ (q, p),$$
$$r^{2n}\xi'_{2n+1}(rq, rp) = \xi'_{2n+1}(q, p),$$
$$r^{2n-1}\zeta'_{2n}(rq, rp) = \zeta'_{2n}\ (q, p).$$

A different method of obtaining these results is mentioned in § 155.

§ 146. The relations in § 143 may be proved directly from the definitions of $\lambda_n(q, p)$, &c., without having recourse to the generating functions of these quantities. For, taking the λ-formula, we have

$$\lambda_{2n+1}(2q, 2p) = A_{2n+1}\left(\frac{1}{2q}\right)\sin\frac{2p\pi}{2q} + A_{2n+1}\left(\frac{2}{2q}\right)\sin\frac{4p\pi}{2q}$$
$$+ A_{2n+1}\left(\frac{3}{2q}\right)\sin\frac{6p\pi}{2q} + \ldots + A_{2n+1}\left(\frac{2q-1}{2q}\right)\sin\frac{(2q-1)\,2p\pi}{2q}$$
$$= A_{2n+1}\left(\frac{1}{2q}\right)\left\{\sin\frac{2p\pi}{2q} - \sin\frac{(2q-1)\,2p\pi}{2q}\right\}$$
$$+ A_{2n+1}\left(\frac{2}{2q}\right)\left\{\sin\frac{4p\pi}{2q} - \sin\frac{(2q-2)\,2p\pi}{2q}\right\}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+ A_{2n+1}\left(\frac{q-1}{2q}\right)\left\{\sin\frac{(q-1)\,2p\pi}{2q} - \sin\frac{(q+1)\,2p\pi}{2q}\right\}$$
$$= (-1)^{p-1}2\left\{A_{2n+1}\left(\frac{1}{2q}\right)\sin\frac{(q-1)p\pi}{q} + A_{2n+1}\left(\frac{2}{2q}\right)\sin\frac{(q-2)p\pi}{q}\right.$$
$$\left. + \ldots + A_{2n+1}\left(\frac{q-1}{2q}\right)\sin\frac{p\pi}{q}\right\},$$

which

$$= 2\left\{A_{2n+1}\left(\frac{1}{2q}\right)\sin\frac{p\pi}{q} + A_{2n+1}\left(\frac{2}{2q}\right)\sin\frac{2p\pi}{q}\right.$$
$$\left. + \ldots + A_{2n+1}\left(\frac{q-1}{2q}\right)\sin\frac{(q-1)\,p\pi}{q}\right\},$$

since $\sin\dfrac{(q-r)\,p\pi}{q} = \sin\left(p\pi - \dfrac{rp\pi}{q}\right) = (-1)^{p-1}\sin\dfrac{rp\pi}{q}$.

Multiplying by 2^{2n} we therefore have

$$2^{2n}\lambda_{2n+1}(2q, 2p) = 2^{2n+1}\left\{A_{2n+1}\left(\frac{1}{2q}\right)\sin\frac{p\pi}{q} + A_{2n+1}\left(\frac{2}{2q}\right)\sin\frac{2p\pi}{q}\right.$$
$$\left. + \ldots + A_{2n+1}\left(\frac{q-1}{2q}\right)\sin\frac{(q-1)p\pi}{q}\right\}.$$

Now

$$2^n A_n\left(\frac{x}{2}\right) = A_n(x) - A_n'(x), \quad (\S\ 1),$$

and therefore

$$2^{2n}\lambda_{2n+1}(2q, 2p) = \left\{A_{2n+1}\left(\frac{1}{q}\right) - A'_{2n+1}\left(\frac{1}{q}\right)\right\}\sin\frac{p\pi}{q}$$
$$+ \left\{A_{2n+1}\left(\frac{2}{q}\right) - A'_{2n+1}\left(\frac{2}{q}\right)\right\}\sin\frac{2p\pi}{q}$$
$$+ \left\{A_{2n+1}\left(\frac{q-1}{q}\right) - A'_{2n+1}\left(\frac{q-1}{q}\right)\right\}\sin\frac{(q-1)p\pi}{q}$$
$$= \lambda_{2n+1}(q, p) - \lambda'_{2n+1}(q, p),$$

which $= \lambda_{2n+1}(q, p)$ or $-\lambda'_{2n+1}(q, p)$ according as p is even or uneven, for $\lambda_{2n+1}(q, p)$ is zero when p is uneven, and $\lambda'_{2n+1}(q, p)$ is zero when p is even (§ 138).

§ 147. It may be remarked that the relations in § 140 may be derived from the expansion-equations by means of the general formulæ

$$\frac{\sin\frac{1}{2}a}{\cosh\frac{1}{2}x - \cos\frac{1}{2}a} - \frac{\sin a}{\cosh x - \cos a} = \frac{2\cosh\frac{1}{2}x\sin\frac{1}{2}a}{\cosh x - \cos a},$$
$$\frac{\sinh\frac{1}{2}x}{\cosh\frac{1}{2}x - \cos\frac{1}{2}a} - \frac{\sinh x}{\cosh x - \cos a} = \frac{2\sinh\frac{1}{2}x\cos\frac{1}{2}a}{\cosh x - \cos a}.$$

Values of the series $\Sigma_{s=-\infty}^{s=\infty}\dfrac{1}{(p+sq)^n}$, &c., §§ 148–155.

§ 148. Expanding $\dfrac{1}{\sinh qt}$ in powers of e^{-qt}, we have

$$\frac{\sinh(q-p)t}{\sinh qt} = \left\{e^{(q-p)t} - e^{-(q-p)t}\right\}\left(e^{-qt} + e^{-3qt} + e^{-5qt} + \&c.\right)$$
$$= e^{-pt} - e^{-(2q-p)t} + e^{-(2q+p)t} - e^{-(4q-p)t} + e^{-(4q+p)t} - \&c.,$$

and therefore, multiplying by t^{2n} and integrating, we find, if $p < 2q$,

$$\int_0^\infty t^{2n} \frac{\sinh (q-p)\,t}{\sinh qt}\,dt = (2n)!\left\{\frac{1}{p^{2n+1}} + \frac{1}{(p-2q)^{2n+1}} + \frac{1}{(p+2q)^{2n+1}} + \frac{1}{(p-4q)^{2n+1}} + \frac{1}{(p+4q)^{2n+1}} + \&c.\right\}.$$

It follows therefore from § 128 that, if p and q are positive integers and $p < 2q$, then the series

$$\frac{1}{p^{2n+1}} + \frac{1}{(p-2q)^{2n+1}} + \frac{1}{(p+2q)^{2n+1}} + \frac{1}{(p-4q)^{2n+1}} + \frac{1}{(p+4q)^{2n+1}} + \&c.$$

$$= (-1)^n \frac{\pi^{2n+1}}{(2n)!}\,\frac{\lambda'_{2n+1}(q, p)}{q} \text{ or } (-1)^{n-1}\frac{\pi^{2n+1}}{(2n)!}\,\frac{\lambda_{2n+1}(q, p)}{q},$$

according as p is uneven or even.

Since each side of this equation is unaltered by substituting $p \pm 2q$ for p, it is clear that the restriction of p being $< 2q$ is unnecessary.

§ 149. In the same manner we find

$$\int_0^\infty t^{2n} \frac{\cosh (q-p)\,t}{\cosh qt}\,dt = (2n)!\left\{\frac{1}{p^{2n+1}} - \frac{1}{(p-2q)^{2n+1}} - \frac{1}{(p+2q)^{2n+1}} + \frac{1}{(p-4q)^{2n+1}} + \frac{1}{(p+4q)^{2n+1}} - \&c.\right\}.$$

The restriction of $p < 2q$ may be removed, since the substitution of $p \pm 2q$ for p merely changes the sign of each side of the equation.

Thus we find that, p and q being any two integers, the series

$$\frac{1}{p^{2n+1}} - \frac{1}{(p-2q)^{2n+1}} - \frac{1}{(p+2q)^{2n+1}} + \frac{1}{(p-4q)^{2n+1}} + \frac{1}{(p+4q)^{2n+1}} - \&c.$$

$$= (-1)^n \frac{\pi^{2n+1}}{(2n)!}\,\frac{\xi'_{2n+1}(q, p)}{q} \text{ or } (-1)^{n-1}\frac{\pi^{2n+1}}{(2n)!}\,\frac{\xi_{2n+1}(q, p)}{q},$$

according as p is uneven or even.

§ 150. The system of four summations so obtained may be written:

$$\text{(i)}\quad \Sigma_{s=-\infty}^{s=\infty}\frac{1}{(p+2sq)^{2n+1}}=(-1)^n\frac{\pi^{2n+1}}{(2n)!}\,\frac{\lambda'_{2n+1}(q,p)}{q}$$

$$\text{or } (-1)^{n-1}\frac{\pi^{2n+1}}{(2n)!}\,\frac{\lambda_{2n+1}(q,p)}{q},$$

$$\text{(ii)}\quad \Sigma_{s=-\infty}^{s=\infty}\frac{1}{(p+2sq)^{2n}}=(-1)^n\frac{\pi^{2n}}{(2n-1)!}\,\frac{\mu'_{2n}(q,p)}{q}$$

$$\text{or } (-1)^{n-1}\frac{\pi^{2n}}{(2n-1)!}\,\frac{\mu_{2n}(q,p)}{q},$$

$$\text{(iii)}\quad \Sigma_{s=-\infty}^{s=\infty}(-1)^s\frac{1}{(p+2sq)^{2n+1}}=(-1)^n\frac{\pi^{2n+1}}{(2n)!}\,\frac{\xi'_{2n+1}(q,p)}{q}$$

$$\text{or } (-1)^{n-1}\frac{\pi^{2n+1}}{(2n)!}\,\frac{\xi_{2n+1}(q,p)}{q},$$

$$\text{(iv)}\quad \Sigma_{s=-\infty}^{s=\infty}(-1)^s\frac{1}{(p+2sq)^{2n}}=(-1)^n\frac{\pi^{2n}}{(2n-1)!}\,\frac{\zeta'_{2n}(q,p)}{q}$$

$$\text{or } (-1)^{n-1}\frac{\pi^{2n}}{(2n-1)!}\,\frac{\zeta_{2n}(q,p)}{q},$$

the first or second value being taken according as p is uneven or even.

§ 151. Putting $2p$ for p, we deduce the general formulæ:

$$\text{(i)}\quad \Sigma_{s=-\infty}^{s=\infty}\frac{1}{(p+sq)^{2n+1}}=(-1)^{n-1}\frac{(2\pi)^{2n+1}}{(2n)!}\,\frac{\lambda_{2n+1}(q,2p)}{q},$$

$$\text{(ii)}\quad \Sigma_{s=-\infty}^{s=\infty}\frac{1}{(p+sq)^{2n}}=(-1)^{n-1}\frac{(2\pi)^{2n}}{(2n-1)!}\,\frac{\mu_{2n}(q,2p)}{q},$$

$$\text{(iii)}\quad \Sigma_{s=-\infty}^{s=\infty}(-1)^s\frac{1}{(p+sq)^{2n+1}}=(-1)^{n-1}\frac{(2\pi)^{2n+1}}{(2n)!}\,\frac{\xi_{2n+1}(q,2p)}{q},$$

$$\text{(iv)}\quad \Sigma_{s=-\infty}^{s=\infty}(-1)^s\frac{1}{(p+sq)^{2n}}=(-1)^{n-1}\frac{(2\pi)^{2n}}{(2n-1)!}\,\frac{\zeta_{2n}(q,2p)}{q}.$$

§ 152. It will be observed that the original results of § 150 may also be readily derived from these formulæ: for, taking

the first equation and supposing $q=2q'$, we see that, omitting the factor $(-1)^{n-1}\dfrac{\pi^{2n+1}}{(2n)!}$, the function which occurs on the right-hand side

$$=\frac{2^{2n+1}\lambda_{2n+1}(2q',\,2p)}{2q'}$$

$$=\frac{\lambda_{2n+1}(q',p)-\lambda'_{2n+1}(q',p)}{q'},\quad (\S\,146),$$

which $=\dfrac{\lambda_{2n+1}(q',p)}{q'}$ or $-\dfrac{\lambda'_{2n+1}(q',p)}{q'}$ according as p is even or uneven (§ 146).

§ 153. It is easy to see (and, indeed, may be regarded as well known) that the values of the series

$$\Sigma_{-\infty}^{\infty}\frac{1}{(p+sq)^n}\ \text{and}\ \Sigma_{-\infty}^{\infty}(-1)^s\frac{1}{(p+sq)^n}$$

are expressed by $\dfrac{\pi^n}{q^n}$ multiplied by a quantity involving only circular functions of $\dfrac{p\pi}{q}$; for, starting with the formula

$$\cot x=\frac{1}{x}+\frac{1}{x-\pi}+\frac{1}{x+\pi}+\frac{1}{x-2\pi}+\frac{1}{x+2\pi}+\&c.,$$

and, replacing x by $\dfrac{p\pi}{q}$, p and q being any quantities (not necessarily integers), we have

$$\frac{\pi}{q}\cot\frac{p\pi}{q}=\frac{1}{p}+\frac{1}{p-q}+\frac{1}{p+q}+\frac{1}{p-2q}+\frac{1}{p+2q}+\&c.,$$

whence, differentiating $(n-1)$ times with respect to p,

$$\Sigma_{-\infty}^{\infty}\frac{1}{(p+sq)^n}=\frac{(-1)^{n-1}}{(n-1)!}\left(\frac{d}{dp}\right)^{n-1}\frac{\pi}{q}\cot\frac{p\pi}{q}.$$

Similarly from

$$\frac{\pi}{q}\cos\frac{p\pi}{q}=\frac{1}{p}-\frac{1}{p-q}-\frac{1}{p+q}+\frac{1}{p-2q}+\frac{1}{p+2q}-\&c.$$

we find

$$\Sigma_{-\infty}^{\infty}(-1)^s\frac{1}{(p+sq)^n}=\frac{(-1)^{n-1}}{(n-1)!}\left(\frac{d}{dp}\right)^{n-1}\frac{\pi}{q}\operatorname{cosec}\frac{p\pi}{q}.$$

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§ 154. These formulæ merely express the values of the series by means of a repeated differentiation of $\cot\dfrac{p\pi}{q}$ and $\operatorname{cosec}\dfrac{p\pi}{q}$; the quantities p and q are, however, unrestricted. The formulæ of § 151 afford actual expressions for the series, but only in the case in which p and q are integers.

§ 155. It may be remarked that, by replacing p and q by rp and rq in the formulæ of § 150, we obtain another proof of the relations connecting $\lambda_{2n+1}(rq, rp)$, &c. which have been given in § 145.

Formulæ involving $A_{2n+1}\left(\dfrac{1}{q}\right)$, $A_{2n+1}\left(\dfrac{2}{q}\right)$, &c., *and* $A'_{2n+1}\left(\dfrac{1}{q}\right)$, $A'_{2n+1}\left(\dfrac{2}{q}\right)$, &c., §§ 156–165.

§ 156. From the expansion-formulæ of § 118, we may derive various general results of the same class as the fundamental theorems

$$A_{2n}\left(\frac{1}{k}\right)+A_{2n}\left(\frac{2}{k}\right)+\ldots+A_{2n}\left(\frac{k-1}{k}\right)=(-1)^n\left(1-\frac{1}{k^{2n-1}}\right)\frac{B_n}{2n},$$

k being any positive integer (*Q.J.*, § 34, p. 21); and

$$A'_{2n}\left(\frac{1}{k}\right)-A'_{2n}\left(\frac{2}{k}\right)+A'_{2n}\left(\frac{3}{k}\right)-\ldots-A'_{2n}\left(\frac{k-1}{k}\right)$$

$$=(-1)^n\left(1-\frac{1}{k^{2n-1}}\right)(2^{2n}-1)\frac{B_n}{2n},$$

k being any positive uneven integer (*Q.J.*, § 190, p. 101).

§ 157. Consider first the expansion

$$\frac{\sin\dfrac{p\pi}{q}}{\cosh x-\cos\dfrac{p\pi}{q}}=-2\Sigma_0^\infty\lambda_{2n+1}(q,p)\frac{(qx)^{2n}}{(2n)!}\text{ or }2\Sigma_0^\infty\lambda'_{2n+1}(q,p)\frac{(qx)^{2n}}{(2n)!},$$

in which

$$\lambda_n(q, p) = A_n\left(\frac{1}{q}\right)\sin\frac{p\pi}{q} + A_n\left(\frac{2}{q}\right)\sin\frac{2p\pi}{q} + \dots$$
$$+ A_n\left(\frac{q-1}{q}\right)\sin\frac{(q-1)p\pi}{q},$$

$$\lambda_n'(q, p) = A_n'\left(\frac{1}{q}\right)\sin\frac{p\pi}{q} + A_n'\left(\frac{2}{q}\right)\sin\frac{2p\pi}{q} + \dots$$
$$+ A_n'\left(\frac{q-1}{q}\right)\sin\frac{(q-1)p\pi}{q},$$

and the first or second value is to be taken according as p is even or uneven.

§ 158. Let $\frac{p}{q} = \frac{1}{2}$, so that $\cos\frac{p\pi}{q} = 0$ and $\sin\frac{p\pi}{q} = 1$. The quantity expanded is therefore $\frac{1}{\cosh x}$; and, since $q = 2p$, q is necessarily even, and we have

$$\lambda_{2n+1}(q, p) = A_{2n+1}\left(\frac{1}{q}\right) - A_{2n+1}\left(\frac{3}{q}\right) + A_{2n+1}\left(\frac{5}{q}\right) - \dots \pm A_{2n+1}\left(\frac{q-1}{q}\right).$$

Now (§ 28) we know that

$$\frac{1}{\cosh x} = 1 - \frac{E_1}{2!}x^2 + \frac{E_2}{4!}x^4 - \frac{E_3}{6!}x^6 + \&c.;$$

whence, equating coefficients, we obtain the theorems:

If $q \equiv 0$, mod. 4, then

$$q^{2n}\left\{A_{2n+1}\left(\frac{1}{q}\right) - A_{2n+1}\left(\frac{3}{q}\right) + A_{2n+1}\left(\frac{5}{q}\right) - \dots\right\} = (-1)^{n-1}\tfrac{1}{2}E_n;$$

and, if $q \equiv 2$, mod. 4, then

$$q^{2n}\left\{A'_{2n+1}\left(\frac{1}{q}\right) - A'_{2n+1}\left(\frac{3}{q}\right) + A'_{2n+1}\left(\frac{5}{q}\right) - \dots\right\} = (-1)^{n}\tfrac{1}{2}E_n.$$

The terms are alternately positive and negative, and the series are to be continued up to and including the term whose argument is $\frac{q-1}{q}$.

§ 159. Let $\frac{p}{q}=\frac{1}{3}$, so that $\cos\frac{p\pi}{q}=\frac{1}{2}$. The quantity expanded is therefore $\frac{\sqrt{3}}{2\cosh x-1}$, and in this case

$$\lambda_{2n+1}(q,p)=\frac{\sqrt{3}}{2}\left\{A_{2n+1}\left(\frac{1}{q}\right)+A_{2n+1}\left(\frac{2}{q}\right)-A_{2n+1}\left(\frac{4}{q}\right)-A_{2n+1}\left(\frac{5}{q}\right)+\ldots\right\};$$

and, since $q=3p$, q is necessarily a multiple of 3.

Comparing this expansion with the formula

$$\frac{1}{2\cosh x-1}=\frac{2}{3}\left\{H_0-\frac{H_1}{2!}x^2+\frac{H_2}{4!}x^4-\&c.\right\},\quad (\S\,29),$$

we obtain the theorems:

If $q\equiv 0$, mod. 6, then

$$q^{2n}\left\{A_{2n+1}\left(\frac{1}{q}\right)+A_{2n+1}\left(\frac{2}{q}\right)-A_{2n+1}\left(\frac{4}{q}\right)-A_{2n+1}\left(\frac{5}{q}\right)+\ldots\right\}$$
$$=(-1)^{n-1}\tfrac{2}{3}H_n,$$

and, if $q\equiv 3$, mod. 6, then

$$q^{2n}\left\{A'_{2n+1}\left(\frac{1}{q}\right)+A'_{2n+1}\left(\frac{2}{q}\right)-A'_{2n+1}\left(\frac{4}{q}\right)-A'_{2n+1}\left(\frac{5}{q}\right)+\ldots\right\}$$
$$=(-1)^{n}\tfrac{2}{3}H_n.$$

Multiples of 3 are excluded from the numerators of the arguments; the terms are alternately positive and negative in pairs; and the argument of the last term is $\frac{q-1}{q}$.

§ 160. Let $\frac{p}{q}=\frac{2}{3}$, so that $\cos\frac{p\pi}{q}=-\frac{1}{2}$. The quantity expanded is $\frac{\sqrt{3}}{2\cosh x+1}$, and

$$\lambda_{2n+1}(q,p)=\frac{\sqrt{3}}{2}\left\{A_{2n+1}\left(\frac{1}{q}\right)-A_{2n+1}\left(\frac{2}{q}\right)+A_{2n+1}\left(\frac{4}{q}\right)-A_{2n+1}\left(\frac{5}{q}\right)+\ldots\right\}.$$

Since $3p=2q$, we see that q must be a multiple of 3, and that p must be even. We do not therefore obtain an A'-theorem in this case.

Comparing the above expansion with the formula (§ 29)

$$\frac{1}{2\cosh x+1}=\tfrac{2}{3}\left\{I_0-\frac{I_1}{2!}a^2+\frac{I_2}{4!}a^4-\frac{I_3}{6!}a^6+\&c.\right\},$$

we obtain the theorem:

If $q\equiv 0$, mod. 3, then

$$q^{2n}\left\{A_{2n+1}\left(\frac{1}{q}\right)-A_{2n+1}\left(\frac{2}{q}\right)+A_{2n+1}\left(\frac{4}{q}\right)-A_{2n+1}\left(\frac{5}{q}\right)+\dots\right\}$$
$$=(-1)^{n-1}\tfrac{2}{3}I_n.$$

Multiples of 3 are excluded from the numerators of the arguments; the terms are alternately positive and negative, and the argument of the last term is $\frac{q-1}{q}$.

§ 161. Other theorems of the same kind may be derived from the expansion

$$\frac{\cosh\frac{1}{2}x\sin\frac{p\pi}{q}}{\cosh x-\cos\frac{p\pi}{2q}}=-\Sigma_0^\infty\xi_{2n+1}(q,p)\frac{(qx)^{2n}}{(2n)!}\text{ or }\Sigma_0^\infty\xi'_{2n+1}(q,p)\frac{(qx)^{2n}}{(2n)!},$$

where

$$\xi_n(q,p)=A_n\left(\frac{1}{2q}\right)\sin\frac{p\pi}{2q}+A_n\left(\frac{3}{2q}\right)\sin\frac{3p\pi}{2q}+\dots$$
$$+A_n\left(\frac{2q-1}{2q}\right)\sin\frac{(2q-1)p\pi}{2q}.$$

§ 162. Let $\frac{p}{q}=\frac{1}{2}$. The quantity expanded is $\frac{1}{\sqrt{2}}\frac{\cosh\frac{1}{2}x}{\cosh x}$, and, from § 31,

$$\frac{\cosh x}{\cosh 2x}=P_0-\frac{P_1}{2!}x^2+\frac{P_2}{4!}x^4-\frac{P_3}{6!}x^6+\&c.$$

We thus find that, if p be even, in which case q is a multiple of 4,

$$(2q)^{2n}\left\{A_{2n+1}\left(\frac{1}{2q}\right)+A_{2n+1}\left(\frac{3}{2q}\right)-A_{2n+1}\left(\frac{5}{2q}\right)-A_{2n+1}\left(\frac{7}{2q}\right)+\dots\right\}$$
$$=(-1)^{n-1}P_n,$$

with a corresponding A'-formula for the case of p uneven.

Replacing $2q$ by q, we have therefore the following theorems:

If $q\equiv 0$, mod. 8, then

$$q^{2n}\left\{A_{2n+1}\left(\frac{1}{q}\right)+A_{2n+1}\left(\frac{3}{q}\right)-A_{2n+1}\left(\frac{5}{q}\right)-A_{2n+1}\left(\frac{7}{q}\right)+\dots\right\}=(-1)^{n-1}P_n;$$

and, if $q\equiv 4$, mod. 8, then

$$q^{2n}\left\{A'_{2n+1}\left(\frac{1}{q}\right)+A'_{2n+1}\left(\frac{3}{q}\right)-A'_{2n+1}\left(\frac{5}{q}\right)-A'_{2n+1}\left(\frac{7}{q}\right)+\dots\right\}=(-1)^{n}P_n.$$

The terms are alternately positive and negative in pairs, and the argument of the last term is $\frac{q-1}{q}$.

§ 163. Let $\frac{p}{q}=\frac{1}{3}$. The quantity expanded is

$$\frac{\cosh\frac{1}{2}x}{2\cosh x-1}=\frac{\cosh^2\frac{1}{2}x}{\cosh\frac{3}{2}x};$$

and, from § 31,

$$\frac{\cosh^2 x}{\cosh 3x}=R_0-\frac{R_1}{2!}x^2+\frac{R_2}{4!}x^4-\frac{R_3}{6!}x^6+\&c.$$

In this case

$$\xi_{2n+1}(q,p)=\tfrac{1}{2}\left\{A_{2n+1}\left(\frac{1}{2q}\right)+2A_{2n+1}\left(\frac{3}{2q}\right)+A_{2n+1}\left(\frac{5}{2q}\right)\right.$$
$$\left.-A_{2n+1}\left(\frac{1}{2q}\right)-2A_{2n+1}\left(\frac{9}{2q}\right)-A_{2n+1}\left(\frac{11}{2q}\right)+\dots\right\};$$

whence, replacing $2q$ by q as in the preceding section, we obtain the theorems:

If $q\equiv 6$, mod. 12, then

$$q^{2n}\left\{A_{2n+1}\left(\frac{1}{q}\right)+2A_{2n+1}\left(\frac{3}{q}\right)+A_{2n+1}\left(\frac{5}{q}\right)-\dots\right\}=(-1)^{n-1}2R_n;$$

and, if $q\equiv 6$, mod. 12, then

$$q^{2n}\left\{A'_{2n+1}\left(\frac{1}{q}\right)+2A'_{2n+1}\left(\frac{3}{q}\right)+A'_{2n+1}\left(\frac{5}{q}\right)-\dots\right\}=(-1)^{n}2R_n.$$

When the numerator of the argument is a multiple of 3 the term has the coefficient 2; the terms are positive or negative in groups of three, and the argument of the last term is $\frac{q-1}{q}$.

§ 164. The value of R_n is $\frac{3^{2n+1}+1}{4}$ (§ 49), and it is evident that these formulæ may be deduced from those of § 158. For, taking the A-formula, we have, from § 158, if $q\equiv 0$, mod. 4,

$$q^{2n}\left\{A_{2n+1}\left(\frac{1}{q}\right)-A_{2n+1}\left(\frac{3}{q}\right)+A_{2n+1}\left(\frac{5}{q}\right)-A_{2n+1}\left(\frac{7}{q}\right)+\ldots\right\}=(-1)^{n-1}\tfrac{1}{2}E_n.$$

Now suppose $q\equiv 0$, mod. 12; then q is divisible by 3, and, by putting $\frac{1}{3}q$ for q, we have

$$q^{2n}\left\{A_{2n+1}\left(\frac{3}{q}\right)-A_{2n+1}\left(\frac{9}{q}\right)+A_{2n+1}\left(\frac{15}{q}\right)-\ldots\right\}=(-1)^{n-1}\tfrac{1}{2}.3^{2n}E_n.$$

By multiplying the second of these eqnations by 3 and adding it to the first, we obtain the formula of § 163.

§ 165. Let $\frac{p}{q}=\frac{2}{3}$. The quantity expanded $=\frac{\sqrt{3}}{2}\cdot\frac{\sinh x}{\sinh\frac{3}{2}x}$, and, from § 36,

$$\frac{\sinh 2x}{\sinh 3x}=\tfrac{1}{3}\left\{J_0-\frac{J_1}{2!}x_2+\frac{J_2}{4!}x^4-\frac{J_3}{6!}x^6+\&c.\right\},$$

whence, proceeding as before, we obtain the theorem:

If $q\equiv 0$, mod. 6, then

$$q^{2n}\left\{A_{2n+1}\left(\frac{1}{q}\right)-A_{2n+1}\left(\frac{5}{q}\right)+A_{2n+1}\left(\frac{7}{q}\right)-A_{2n+1}\left(\frac{11}{q}\right)+\ldots\right\}=(-1)^{n-1}\tfrac{1}{3}J_n.$$

Multiples of 3 do not occur in the numerators of the arguments; the terms are alternately positive and negative, and the argument of the last term is $\frac{q-1}{q}$.

This result is, however, derivable by addition from the A-formulæ of §§ 159 and 160, since $I_n+H_n=J_n$ (§ 30).

Since p is even, there is no corresponding A'-theorem.

Formulæ involving $A_n\left(\frac{1}{q}\right)$, $A_{2n}\left(\frac{2}{q}\right)$, &c., *and*

$A'_{2n}\left(\frac{1}{q}\right)$, $A'_{2n}\left(\frac{2}{q}\right)$, &c., §§ 166–173.

§166. The independent A- and A'-formulæ with even suffixes are less numerous, as most of those given by the expansions may be deduced from the two fundamental theorems quoted in § 156.

In the expansion

$$\frac{\sinh x}{\cosh x - \cos\frac{p\pi}{q}} = 2\Sigma_1^\infty \mu_{2n}(q, p)\frac{(qx)^{2n-1}}{(2n-1)!}$$

$$\text{or } -2\Sigma_1^\infty \mu'_{2n}(q, p)\frac{(qx)^{2n-1}}{(2n-1)!},$$

where

$$\mu_n(q,p) = A_n\left(\frac{1}{q}\right)\cos\frac{p\pi}{q} + A_n\left(\frac{2}{q}\right)\cos\frac{2p\pi}{q} + \ldots + A_n\left(\frac{q}{q}\right)\cos\frac{2p\pi}{q},$$

let $\frac{p}{q} = 1$. The quantity expanded is $\tanh\frac{1}{2}x$, and we obtain the theorem:

If q is even, then

$$q^{2n-1}\left\{A_{2n}\left(\frac{1}{q}\right) - A_{2n}\left(\frac{2}{q}\right) + A_{2n}\left(\frac{3}{q}\right) - \ldots - A_{2n}\left(\frac{q}{q}\right)\right\} = (-1)^n(2^{2n}-1)\frac{B_n}{2n};$$

if q is uneven, then

$$q^{2n-1}\left\{A'_{2n}\left(\frac{1}{q}\right) - A'_{2n}\left(\frac{2}{q}\right) + A'_{2n}\left(\frac{3}{q}\right) - \ldots + A'_{2n}\left(\frac{q}{q}\right)\right\}$$

$$= (-1)^{n-1}(2^{2n}-1)\frac{B_n}{2n}.$$

The first of these results is readily deduced from the A-formula of § 156; and the second is equivalent to the A'-formula of the same section, since

$$A'_{2n}(0) = (-1)^n(2^{2n}-1)\frac{B_n}{2n}, \quad (\S\,70).$$

§ 167. If we put $\frac{p}{q}=\frac{1}{2}$, the quantity expanded is $\tanh x$, and the resulting formulæ are equivalent to those obtained in the preceding section.

§ 168. When $\frac{p}{q}=\frac{1}{3}$ and $\frac{p}{q}=\frac{2}{3}$, the quantity expanded becomes

$$\frac{2\sinh x}{2\cosh x-1} \text{ and } \frac{2\sinh x}{2\cosh x+1}$$

respectively.

Now
$$\frac{4\sinh 2x}{2\cosh 2x-1}=3\tanh 3x-\tanh x,$$

and
$$\frac{4\sinh 2x}{2\cosh 2x+1}=3\coth 3x-\coth x;$$

whence we find

$$\frac{2\sinh x}{2\cosh x-1}=\Sigma_1^\infty(-1)^{n-1}\frac{(2^{2n}-1)(3^{2n}-1)B_n}{(2n)!}x^{2n-1},$$

$$\frac{2\sinh x}{2\cosh x+1}=\Sigma_1^\infty(-1)^{n-1}\frac{(3^{2n}-1)B_n}{(2n)!}x^{2n-1}.$$

§ 169. We thus obtain the theorems:

(i) If $q\equiv 0$, mod. 6, then

$$q^{2n-1}\left\{A_{2n}\left(\frac{1}{q}\right)-A_{2n}\left(\frac{2}{q}\right)-2A_{2n}\left(\frac{3}{q}\right)-A_{2n}\left(\frac{4}{q}\right)+A_{2n}\left(\frac{5}{q}\right)\right.$$
$$\left.+2A_{2n}\left(\frac{6}{q}\right)+\&c.\right\}=(-1)^{n-1}(2^{2n}-1)(3^{2n}-1)\frac{B_n}{2n};$$

(ii) If $q\equiv 3$, mod. 6, then

$$q^{2n-1}\left\{A'_{2n}\left(\frac{1}{q}\right)-A'_{2n}\left(\frac{2}{q}\right)-2A'_{2n}\left(\frac{3}{q}\right)-A'_{2n}\left(\frac{4}{q}\right)+A'_{2n}\left(\frac{5}{q}\right)\right.$$
$$\left.+2A'_{2n}\left(\frac{6}{q}\right)+\&c.\right\}=(-1)^{n}(2^{2n}-1)(3^{2n}-1)\frac{B_n}{2n};$$

(iii) If $q\equiv 0$, mod. 3, then

$$q^{2n-1}\left\{A_{2n}\left(\frac{1}{q}\right)+A_{2n}\left(\frac{2}{q}\right)-2A_{2n}\left(\frac{3}{q}\right)+A_{2n}\left(\frac{4}{q}\right)+A_{2n}\left(\frac{5}{q}\right)\right.$$
$$\left.-2A_{2n}\left(\frac{6}{q}\right)+\&c.\right\}=(-1)^{n}(3^{2n}-1)\frac{B_n}{2n},$$

In all three formulæ the terms have the coefficient 2 when the numerator of the argument is divisible by 3; and the argument of the last term is unity. In (i) and (ii) the terms after the first are negative and positive in threes; in (iii) the terms having the coefficient 2 are negative.

These formulæ may evidently be derived from those in § 156.

§ 170. Consider now the expansion

$$\frac{\sinh \frac{1}{2}x \cos \dfrac{p\pi}{2q}}{\cosh x - \cos \dfrac{p\pi}{q}} = \Sigma_1^\infty \zeta_{2n}(q, p) \frac{(qx)^{2n-1}}{(2n-1)!}$$

$$\text{or } -\Sigma_1^\infty \zeta'_{2n}(q, p) \frac{(qx)^{2n-1}}{(2n-1)!},$$

where

$$\zeta_n(q, p) = A_n\left(\frac{1}{2q}\right) \cos\frac{p\pi}{2q} + A_n\left(\frac{3}{2q}\right) \cos\frac{3p\pi}{2q} + \dots$$
$$+ A_n\left(\frac{2q-1}{2q}\right) \cos\frac{(2q-1)p\pi}{2q}.$$

Putting $\dfrac{p}{q} = \frac{1}{2}$, and comparing with the formula (§ 38),

$$\frac{\cosh x}{\cosh 2x} = P_0 - \frac{P_1}{2!}x^2 + \frac{P_2}{4!}x^4 - \frac{P_3}{6!}x^6 + \&c.,$$

we obtain the theorems:

If $q \equiv 0$, mod. 8, then

$$q^{2n-1}\left\{A_{2n}\left(\frac{1}{q}\right) - A_{2n}\left(\frac{3}{q}\right) - A_{2n}\left(\frac{5}{q}\right) + A_{2n}\left(\frac{7}{q}\right) - \dots\right\} = (-1)^{n-1} Q_n;$$

if $q \equiv 4$, mod. 8, then

$$q^{2n-1}\left\{A'_{2n}\left(\frac{1}{q}\right) - A'_{2n}\left(\frac{3}{q}\right) - A'_{2n}\left(\frac{5}{q}\right) + A'_{2n}\left(\frac{7}{q}\right) - \dots\right\} = (-1)^n Q_n.$$

The terms after the first are alternately negative and positive in pairs, and the argument of the last term is $\dfrac{q-1}{q}$.

§ 171. Let $\dfrac{p}{q} = \frac{1}{3}$. The quantity expanded is

$$\frac{\sqrt{3} \sinh \frac{1}{2}x}{2 \cosh x - 1} = \frac{\sqrt{3}}{2} \frac{\sinh x}{\cosh \frac{3}{2}x};$$

and, from § 38,

$$\frac{\sinh 2x}{\cosh 3x} = 2\left\{T_1 x - \frac{T_2}{3!}x^3 + \frac{T_3}{5!}x^5 - \&c.\right\};$$

whence we obtain the theorems:

If $q \equiv 0$, mod. 12, then

$$q^{2n-1}\left\{A_{2n}\left(\frac{1}{q}\right) - A_{2n}\left(\frac{5}{q}\right) - A_{2n}\left(\frac{7}{q}\right) + A_{2n}\left(\frac{11}{q}\right) + \ldots\right\} = (-1)^{n-1} 2T_n;$$

if $q \equiv 6$, mod. 12, then

$$q^{2n-1}\left\{A'_{2n}\left(\frac{1}{q}\right) - A'_{2n}\left(\frac{5}{q}\right) - A'_{2n}\left(\frac{7}{q}\right) + A'_{2n}\left(\frac{11}{q}\right) - \ldots\right\} = (-1)^n 2T_n.$$

The terms after the first are alternately negative and positive in pairs, and the argument of the last term is $\frac{q-1}{q}$.

§ 172. Let $\frac{p}{q} = \frac{1}{6}$. The quantity expanded becomes

$$\frac{\sinh \frac{1}{2}x}{2\cosh x + 1} = \frac{\sinh^2 \frac{1}{2}x}{\sinh \frac{3}{2}x}.$$

Now

$$\frac{4\sinh^2 x}{\sinh 3x} = \frac{1}{\sinh x} - \frac{3}{\sinh 3x};$$

whence we find

$$\frac{2\sinh^2 x}{\sinh 3x} = \Sigma_1^\infty (-1)^{n-1}\frac{(2^{2n-1}-1)(3^{2n}-1)B_n}{(2n)!}x^{2n-1}.$$

We thus obtain the theorem:

If $q \equiv 0$, mod. 6, then

$$q^{2n-1}\left\{A_{2n}\left(\frac{1}{q}\right) - 2A_{2n}\left(\frac{3}{q}\right) + A_{2n}\left(\frac{5}{q}\right) + A_{2n}\left(\frac{7}{q}\right) - 2A_{2n}\left(\frac{9}{q}\right)\right.$$
$$\left. + A_{2n}\left(\frac{11}{q}\right) + \&c.\right\} = (-1)^{n-1}(2^{2n-1}-1)(3^{2n}-1)\frac{B_n}{2n}.$$

The terms in which is the numerator of the argument is divisible by 3 have the coefficient 2 and are negative, and the argument of the last term is $\frac{q-1}{q}$.

This result may evidently be derived from the A-formula in § 156.

§ 173. If $\frac{p}{q}=2$, the quantity expanded is $-\frac{1}{2\sinh\frac{1}{2}x}$, and we obtain the theorem:

If q be even, then

$$q^{2n-1}\left\{A_{2n}\left(\frac{1}{q}\right)+A_{2n}\left(\frac{3}{q}\right)+A_{2n}\left(\frac{5}{q}\right)+\ldots+A_{2n}\left(\frac{q-1}{q}\right)\right\}$$
$$=(-1)^n(2^{2n-1}-1)\frac{B_n}{2n}.$$

This result also may be derived from the A-formula in § 156.

Résumé and discussion of the preceding formulæ, §§ 174–182.

§ 174. Collecting and arranging the formulæ in §§ 157–173, and omitting those in which the value of the series depends upon Bernoullian numbers (as they may be derived from the fundamental formulæ in § 156), we obtain the following system of formulæ involving the function A_{2n+1}.

(i) $$q^{2n}\left\{A_{2n+1}\left(\frac{1}{q}\right)-A_{2n+1}\left(\frac{3}{q}\right)+A_{2n+1}\left(\frac{5}{q}\right)-A_{2n+1}\left(\frac{7}{q}\right)+\ldots\right\}$$
$$=(-1)^{n-1}\tfrac{1}{2}E_n \text{ if } q\equiv 0, \text{ mod. } 4, \quad (\S\ 158),$$

(ii) $$q^{2n}\left\{A_{2n+1}\left(\frac{1}{q}\right)+A_{2n+1}\left(\frac{3}{q}\right)-A_{2n+1}\left(\frac{5}{q}\right)-A_{2n+1}\left(\frac{7}{q}\right)+\ldots\right\}$$
$$=(-1)^{n-1}P_n \text{ if } q\equiv 0, \text{ mod. } 8, \quad (\S\ 162),$$

(iii) $$q^{2n}\left\{A_{2n+1}\left(\frac{1}{q}\right)+A_{2n+1}\left(\frac{2}{q}\right)-A_{2n+1}\left(\frac{4}{q}\right)-A_{2n+1}\left(\frac{5}{q}\right)+\ldots\right\}$$
$$=(-1)^{n-1}\tfrac{2}{3}H_n \text{ if } q\equiv 0, \text{ mod. } 6, \quad (\S\ 159),$$

(iv) $$q^{2n}\left\{A_{2n+1}\left(\frac{1}{q}\right)-A_{2n+1}\left(\frac{2}{q}\right)+A_{2n+1}\left(\frac{4}{q}\right)-A_{2n+1}\left(\frac{5}{q}\right)+\ldots\right\}$$
$$=(-1)^{n-1}\tfrac{2}{3}I_n \text{ if } q\equiv 0, \text{ mod. } 3, \quad (\S\ 160),$$

(v) $$q^{2n}\left\{A_{2n+1}\left(\frac{1}{q}\right)-A_{2n+1}\left(\frac{5}{q}\right)+A_{2n+1}\left(\frac{7}{q}\right)-A_{2n+1}\left(\frac{11}{q}\right)+\ldots\right\}$$
$$=(-1)^{n-1}\tfrac{1}{3}J_n \text{ if } q\equiv 0, \text{ mod. } 6, \quad (\S\ 165).$$

The last three formulæ are not independent, any two being deducible from the third, since $I_n+H_n=J_n$ and $2H_n-J_n=2^{2n+1}I_n$, (§ 30).

By adding (i) and (ii), we find

$$\text{(vi)}\quad q^{2n}\left\{A_{2n+1}\left(\frac{1}{q}\right)-A_{2n+1}\left(\frac{7}{q}\right)+A_{2n+1}\left(\frac{9}{q}\right)-A_{2n+1}\left(\frac{15}{q}\right)+\dots\right\}$$

$$=(-1)^{n-1}\frac{E_n+2P_n}{4}\ \text{if}\ q\equiv 0,\ \text{mod. } 8.$$

The formula involving R_n (§ 163) is omitted as it is derivable from (i), and the law of the series is not of a fundamental character.

§ 175. Omitting as before the results depending upon Bernoullian numbers, the formulæ involving the function A_{2n} are:

$$\text{(i)}\quad q^{2n-1}\left\{A_{2n}\left(\frac{1}{q}\right)-A_{2n}\left(\frac{3}{q}\right)-A_{2n}\left(\frac{5}{q}\right)+A_{2n}\left(\frac{7}{q}\right)+\dots\right\}$$

$$=(-1)^{n-1}Q_n\ \text{if}\ q\equiv 0,\ \text{mod. } 8,\quad (\S\,170),$$

$$\text{(ii)}\quad q^{2n-1}\left\{A_{2n}\left(\frac{1}{q}\right)-A_{2n}\left(\frac{5}{q}\right)-A_{2n}\left(\frac{7}{q}\right)+A_{2n}\left(\frac{11}{q}\right)+\dots\right\}$$

$$=(-1)^{n-1}2T_n\ \text{if}\ q\equiv 0,\ \text{mod. } 12,\quad (\S\,171).$$

§ 176. The formulæ involving A'_{2n+1} and not depending upon Bernoullian numbers are:

$$\text{(i)}\quad q^{2n}\left\{A'_{2n+1}\left(\frac{1}{q}\right)-A'_{2n+1}\left(\frac{3}{q}\right)+A'_{2n+1}\left(\frac{5}{q}\right)-A'_{2n+1}\left(\frac{7}{q}\right)+\dots\right\}$$

$$=(-1)^{n}\tfrac{1}{2}E_n\ \text{if}\ q\equiv 2,\ \text{mod. } 4,\quad (\S\,158),$$

$$\text{(ii)}\quad q^{2n}\left\{A'_{2n+1}\left(\frac{1}{q}\right)+A'_{2n+1}\left(\frac{3}{q}\right)-A'_{2n+1}\left(\frac{5}{q}\right)-A'_{2n+1}\left(\frac{7}{q}\right)+\dots\right\}$$

$$=(-1)^{n}P_n\ \text{if}\ q\equiv 4,\ \text{mod. } 8,\quad (\S\,162),$$

$$\text{(iii)}\quad q^{2n}\left\{A'_{2n+1}\left(\frac{1}{q}\right)+A'_{2n+1}\left(\frac{2}{q}\right)-A'_{2n+1}\left(\frac{4}{q}\right)-A'_{2n+1}\left(\frac{5}{q}\right)+\dots\right\}$$

$$=(-1)^{n}\tfrac{2}{3}H_n\ \text{if}\ q\equiv 3,\ \text{mod. } 6,\quad (\S\,159).$$

Adding (i) and (ii), we have

$$\text{(iv)}\quad q^{2n}\left\{A'_{2n+1}\left(\frac{1}{q}\right)-A'_{2n+1}\left(\frac{7}{q}\right)+A'_{2n+1}\left(\frac{9}{q}\right)-A'_{2n+1}\left(\frac{15}{q}\right)+\dots\right\}$$

$$=(-1)^{n}\frac{E_n+2P_n}{4}\ \text{if}\ q\equiv 4,\ \text{mod. } 8.$$

The R-formula (§ 163) is omitted as in § 174.

§ 177. The similar formulæ involving A'_{2n} are:

$$\text{(i)}\quad q^{2n-1}\left\{A'_{2n}\left(\frac{1}{q}\right)-A'_{2n}\left(\frac{3}{q}\right)-A'_{2n}\left(\frac{5}{q}\right)+A'_{2n}\left(\frac{7}{q}\right)+\ldots\right\}$$
$$=(-1)^n Q_n \text{ if } q\equiv 4,\ \text{mod. } 8, \qquad (\S\,170),$$

$$\text{(ii)}\quad q^{2n-1}\left\{A'_{2n}\left(\frac{1}{q}\right)-A'_{2n}\left(\frac{5}{q}\right)-A'_{2n}\left(\frac{7}{q}\right)+A'_{2n}\left(\frac{11}{q}\right)-\ldots\right\}$$
$$=(-1)^{n-1}2T_n \text{ if } q\equiv 6,\ \text{mod. } 12, \qquad (\S\,171).$$

§ 178. Since

$$8^{2n+1}A_{2n+1}(\tfrac{1}{8})-4^{2n+1}A_{2n+1}(\tfrac{1}{4})=(-1)^{n+1}\,2P_n \quad (Q.\,J.,\ \S\,109,\ \text{p. } 60),$$

we have $\qquad 8^{2n+1}A_{2n+1}(\tfrac{1}{8})=(-1)^{n+1}(E_n+2P_n).$

Thus the right-hand member of equation (vi) of § 174 is equal to $2.8^{2n}A_{2n+1}(\tfrac{1}{8})$.

§ 179. Also we have

$$(12)^{2n+1}A_{2n+1}(\tfrac{1}{12})=6^{2n+1}A_{2n+1}(\tfrac{1}{6})+4^{2n+1}A_{2n+1}(\tfrac{1}{4})+(-1)^{n+1}\,2R_n$$
$$(Q.\,J.,\ \S\,132,\ \text{p. } 70),$$

whence $\qquad (12)^{2n+1}A_{2n+1}(\tfrac{1}{12})=J_n+\tfrac{3}{2}(3^{2n}+1)E_n.$

Thus, from (i) and (v) of § 174, we find that, if $q\equiv 0$, mod. 12, then

$$q^{2n}\left\{A_{2n+1}\left(\frac{1}{q}\right)-A_{2n+1}\left(\frac{11}{q}\right)+A_{2n+1}\left(\frac{13}{q}\right)-A_{2n+1}\left(\frac{23}{q}\right)+\ldots\right\}$$
$$=2.(12)^{2n}A_{2n+1}(\tfrac{1}{12}).$$

§ 180. We thus have obtained the group of formulæ:

$$\text{(i)}\quad q^{2n}\left\{A_{2n+1}\left(\frac{1}{q}\right)-A_{2n+1}\left(\frac{3}{q}\right)+A_{2n+1}\left(\frac{5}{q}\right)-A_{2n+1}\left(\frac{7}{q}\right)+\ldots\right\}$$
$$=2.4^{2n}A_{2n+1}(\tfrac{1}{4}),$$

$$\text{(ii)}\quad q^{2n}\left\{A_{2n+1}\left(\frac{1}{q}\right)-A_{2n+1}\left(\frac{5}{q}\right)+A_{2n+1}\left(\frac{7}{q}\right)-A_{2n+1}\left(\frac{11}{q}\right)+\ldots\right\}$$
$$=2.6^{2n}A_{2n+1}(\tfrac{1}{6}),$$

$$\text{(iii)}\quad q^{2n}\left\{A_{2n+1}\left(\frac{1}{q}\right)-A_{2n+1}\left(\frac{7}{q}\right)+A_{2n+1}\left(\frac{9}{q}\right)-A_{2n+1}\left(\frac{15}{q}\right)+\ldots\right\}$$
$$=2.8^{2n}A_{2n+1}(\tfrac{1}{8}),$$

$$\text{(iv)}\quad q^{2n}\left\{A_{2n+1}\left(\frac{1}{q}\right)-A_{2n+1}\left(\frac{11}{q}\right)+A_{2n+1}\left(\frac{13}{q}\right)-A_{2n+1}\left(\frac{23}{q}\right)+\ldots\right\}$$
$$=2.(12)^{2n}A_{2n+1}(\tfrac{1}{12});$$

in which respectively (i) $q\equiv 0$, mod. 4; (ii) $q\equiv 0$, mod. 6; (iii) $q\equiv 0$, mod. 8; (iv) $q\equiv 0$, mod. 12. The numerators of the arguments $\equiv 1$ or $i-1$, mod. i, where i has the respective values 4, 6, 8, 12.

§ 181. In the case of the function A_{2n}, we have

$$8^{2n}A_{2n}(\tfrac{1}{8})=4^{2n}A_{2n}(\tfrac{1}{4})+(-1)^{n-1}2Q_n \quad (Q.\,J.,\ \S\,119,\ \text{p. }64)$$
$$=(-1)^{n-1}2\left\{Q_n-(2^{2n-1}-1)\frac{B_n}{2n}\right\}.$$

Also

$$(12)^{2n}A_{2n}(\tfrac{1}{12})=6^{2n}A_{2n}(\tfrac{1}{6})+(-1)^{n-1}6T_n \quad (Q.\,J.,\ \S\,143,\ \text{p. }75.)$$
$$=(-1)^{n-1}6\left\{T_n+(2^{2n-1}-1)(3^{2n-1}-1)\frac{B_n}{2n}\right\}.$$

§ 182. Corresponding to the group of formulæ in § 178, we find

$$\text{(i)}\quad q^{2n}\left\{A_{2n}\left(\frac{1}{q}\right)+A_{2n}\left(\frac{3}{q}\right)+A_{2n}\left(\frac{5}{q}\right)+A_{2n}\left(\frac{7}{q}\right)+\ldots\right\}$$
$$=2.4^{2n-1}A_{2n}(\tfrac{1}{4}),$$

$$\text{(ii)}\quad q^{2n}\left\{A_{2n}\left(\frac{1}{q}\right)+A_{2n}\left(\frac{5}{q}\right)+A_{2n}\left(\frac{7}{q}\right)+A_{2n}\left(\frac{11}{q}\right)+\ldots\right\}$$
$$=2.6^{2n-1}A_{2n}(\tfrac{1}{6}),$$

$$\text{(iii)}\quad q^{2n}\left\{A_{2n}\left(\frac{1}{q}\right)+A_{2n}\left(\frac{7}{q}\right)+A_{2n}\left(\frac{9}{q}\right)+A_{2n}\left(\frac{15}{q}\right)+\ldots\right\}$$
$$=2.8^{2n-1}A_{2n}(\tfrac{1}{8}),$$

$$\text{(iv)}\quad q^{2n}\left\{A_{2n}\left(\frac{1}{q}\right)+A_{2n}\left(\frac{11}{q}\right)+A_{2n}\left(\frac{13}{q}\right)+A_{2n}\left(\frac{23}{q}\right)+\ldots\right\}$$
$$=2.12^{2n-1}A_{2n}(\tfrac{1}{12});$$

in which, respectively, (i) $q \equiv 0$, mod. 2; (ii) $q \equiv 0$, mod. 6; (iii) $q \equiv 0$, mod. 8; (iv) $q \equiv 0$, mod. 12. The arguments are the same as in § 178, but all the terms are positive, instead of being positive and negative alternately.

General formulæ of the same kind, §§ 183–207.

§ 183. These results point to the existence of more general formulæ of the same kind, and it will now be shown that such is the case.

Taking the general formula

$$k^{n-1}\left\{A_n(x) + A_n\left(x+\frac{1}{k}\right) + A_n\left(x+\frac{2}{k}\right) + \ldots + A_n\left(x+\frac{k-1}{k}\right)\right\}$$
$$= A_n(kx) \quad (Q.\,J., \text{ § } 34, \text{ p. } 21),$$

let $x = \dfrac{1}{ik}$, i being any positive integer, and put $q = ik$. Then

$$\text{(i)} \quad q^{n-1}\left\{A_n\left(\frac{1}{q}\right) + A_n\left(\frac{i+1}{q}\right) + A_n\left(\frac{2i+1}{q}\right) + \ldots + A_n\left(\frac{q-i+1}{q}\right)\right.$$
$$= i^{n-1} A_n\left(\frac{1}{i}\right).$$

In this equation q is any number $\equiv 0$, mod. i, and the numerators are the numbers $< q$ and $\equiv 1$, mod. i.

Putting now $x = \dfrac{i-1}{ik}$, we find

$$\text{(ii)} \quad q^{n-1}\left\{A_n\left(\frac{i-1}{q}\right) + A_n\left(\frac{2i-1}{q}\right) + A_n\left(\frac{3i-1}{q}\right) + \ldots + A_n\left(\frac{q-1}{q}\right)\right.$$
$$= i^{n-1} A_n\left(\frac{i-1}{q}\right) = (-1)^n i^{n-1} A_n\left(\frac{1}{i}\right),$$

where $q \equiv 0$, mod. i, and the numerators are the numbers $< q$ and $\equiv i-1$, mod. i.

The groups of results in §§ 180 and 178 are particular cases of the equations obtained by addition and subtraction from these two formulæ. It will be noticed that the separate formulæ (i) and (ii) are more simple and interesting than the results obtained by combining them.

§ 184. The most general formula of this class may be obtained by putting $x = \frac{r}{ik}$, r being a positive integer. We thus find that, if $q \equiv 0$, mod. i, then

$$q^{n-1}\left\{A_n\left(\frac{r}{q}\right) + A_n\left(\frac{i+r}{q}\right) + A_n\left(\frac{2i+r}{q}\right) + \ldots + A_n\left(\frac{q-i+r}{q}\right)\right\}$$
$$= i^{n-1} A_n\left(\frac{r}{i}\right),$$

the numerators being the numbers $< q$ and $\equiv r$, mod. i.

The expansion on the left-hand side of this equation is therefore such that it has the same value whatever multiple q may be of i.

§ 185. Similar results relating to the function A'_n may be derived from the general formula

$$k^{n-1}\left\{A'_n(x) - A'_n\left(x+\frac{1}{k}\right) + A'_n\left(x+\frac{2}{k}\right) - \ldots + A'_n\left(x+\frac{k-1}{k}\right)\right\}$$
$$= A'_n(kx) \quad (Q.\ J.,\ \S\ 189,\ \text{p. } 101),$$

where k is any positive uneven integer.

Putting, as before, ik for k and $q = ik$, we find

$$q^{n-1}\left\{A'_n\left(\frac{1}{q}\right) - A'_n\left(\frac{i+1}{q}\right) + A'_n\left(\frac{2i+1}{q}\right) - \ldots + A'_n\left(\frac{q-i+1}{q}\right)\right\}$$
$$= i^{n-1} A'_n\left(\frac{1}{i}\right).$$

Since k is uneven, q must be an uneven multiple of i, that is, $q \equiv i$, mod. $2i$.

In general, if $q \equiv i$, mod. $2i$, and r is a positive integer, then

$$q^{n-1}\left\{A'_n\left(\frac{r}{q}\right) - A'_n\left(\frac{i+r}{q}\right) + A'_n\left(\frac{2i+r}{q}\right) - \ldots + A'_n\left(\frac{q-i+r}{q}\right)\right\}$$
$$= i^{n-1} A'_n\left(\frac{r}{i}\right):$$

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§ 186. It can be shewn that, if k be any even number,

$$k^{n-1}\left\{A_n(x)-A_n\left(x+\frac{1}{k}\right)+A_n\left(x+\frac{2}{k}\right)-\ldots-A_n\left(x+\frac{k-1}{k}\right)\right\},$$
$$=-A'_n(kx).*$$

We thus find, as in the preceding section, that if $q\equiv 0$, mod. $2i$, then

$$q^{n-1}\left\{A_n\left(\frac{1}{q}\right)-A_n\left(\frac{i+1}{q}\right)+A_n\left(\frac{2i+1}{q}\right)-\ldots-A_n\left(\frac{q-i+1}{q}\right)\right\}$$
$$=-i^{n-1}A'_n\left(\frac{1}{i}\right),$$

and that, in general, if $q\equiv 0$, mod. $2i$, and r is any positive integer, then

$$q^{n-1}\left\{A_n\left(\frac{r}{q}\right)-A_n\left(\frac{i+r}{q}\right)+A_n\left(\frac{2i+r}{q}\right)-\ldots-A_n\left(\frac{q-i+r}{q}\right)\right\}$$
$$=-i^{n-1}A'_n\left(\frac{1}{i}\right).$$

§ 187. Omitting results in which the right-hand side is expressed by Bernoullian numbers, we deduce from (i) of § 181, by putting $i=3$, 4, 6, the group of formulæ

(i) $$q^{2n}\left\{A_{2n+1}\left(\frac{1}{q}\right)+A_{2n+1}\left(\frac{4}{q}\right)+A_{2n+1}\left(\frac{7}{q}\right)+\ldots+A_{2n+1}\left(\frac{q-2}{q}\right)\right\}$$
$$=(-1)^{n-1}\tfrac{1}{8}I_n,$$

(ii) $$q^{2n}\left\{A_{2n+1}\left(\frac{1}{q}\right)+A_{2n+1}\left(\frac{5}{q}\right)+A_{2n+1}\left(\frac{9}{q}\right)+\ldots+A_{2n+1}\left(\frac{q-3}{q}\right)\right\}$$
$$=(-1)^{n-1}\tfrac{1}{4}E_n,$$

(iii) $$q^{2n}\left\{A_{2n+1}\left(\frac{1}{q}\right)+A_{2n+1}\left(\frac{7}{q}\right)+A_{2n+1}\left(\frac{13}{q}\right)+\ldots+A_{2n+1}\left(\frac{q-5}{q}\right)\right\}$$
$$=(-1)^{n-1}\tfrac{1}{8}J_n.$$

In (i) $q\equiv 0$, mod. 3; in (ii) $q\equiv 0$, mod. 4; in (iii) $q\equiv 0$, mod. 6.

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* This may be proved as in *Q.J.* § 34, p. 21, or § 189, p. 100. It may be mentioned here that the result given in *Q.J.* § 194, p. 102 is erroneous, except when $c=1$.

§ 188. From § 186, by putting $i=2, 3, 4, 6$, and taking the suffixes uneven, we obtain the group:

(i) $$q^{2n}\left\{A_{2n+1}\left(\frac{1}{q}\right)-A_{2n+1}\left(\frac{3}{q}\right)+A_{2n+1}\left(\frac{5}{q}\right)-\ldots-A_{2n+1}\left(\frac{q-1}{q}\right)\right\}$$
$$=(-1)^{n-1}\tfrac{1}{2}E_n,$$

(ii) $$q^{2n}\left\{A_{2n+1}\left(\frac{1}{q}\right)-A_{2n+1}\left(\frac{4}{q}\right)+A_{2n+1}\left(\frac{7}{q}\right)-\ldots-A_{2n+1}\left(\frac{q-2}{q}\right)\right\}$$
$$=(-1)^{n-1}\tfrac{1}{3}H_n,$$

(iii) $$q^{2n}\left\{A_{2n+1}\left(\frac{1}{q}\right)-A_{2n+1}\left(\frac{5}{q}\right)+A_{2n+1}\left(\frac{9}{q}\right)-\ldots-A_{2n+1}\left(\frac{q-3}{q}\right)\right\}$$
$$=(-1)^{n-1}\tfrac{1}{2}P_n,$$

(iv) $$q^{2n}\left\{A_{2n+1}\left(\frac{1}{q}\right)-A_{2n+1}\left(\frac{7}{q}\right)+A_{2n+1}\left(\frac{13}{q}\right)-\ldots-A_{2n+1}\left(\frac{q-5}{q}\right)\right\}$$
$$=(-1)^{n-1}\frac{3^{2n}+1}{4}E_n.$$

In (i) $q\equiv 0$, mod. 4; in (ii) $q\equiv 0$, mod. 6; in (iii) $q\equiv 0$, mod. 8; in (iv), $q\equiv 0$, mod. 12.

§ 189. From § 186, we also obtain the following formulæ in which the suffixes are even:

(i) $$q^{2n-1}\left\{A_{2n}\left(\frac{1}{q}\right)-A_{2n}\left(\frac{5}{q}\right)+A_{2n}\left(\frac{9}{q}\right)-\ldots-A_{2n}\left(\frac{q-3}{q}\right)\right\}$$
$$=(-1)^{n-1}\tfrac{1}{2}Q_n,$$

(ii) $$q^{2n-1}\left\{A_{2n}\left(\frac{1}{q}\right)-A_{2n}\left(\frac{7}{q}\right)+A_{2n}\left(\frac{13}{q}\right)-\ldots-A_{2n}\left(\frac{q-5}{q}\right)\right\}$$
$$=(-1)^{n-1}T_n.$$

In (i) $q\equiv 0$, mod. 8; in (ii) $q\equiv 0$, mod. 12.

§ 190. It is unnecessary to write down the corresponding groups in which the argument of the first term is $\dfrac{i-1}{q}$ instead of $\dfrac{1}{q}$ as the form of q is the same, and the only difference that occurs is in the sign of the right-hand side which is changed in the groups corresponding to §§ 187 and 189. Thus for

example corresponding to (i) of § 187, we have, if $q \equiv 0$, mod. 3,

$$\text{(i)}\quad q^{2n}\left\{A_{2n+1}\left(\frac{2}{q}\right)+A_{2n+1}\left(\frac{5}{q}\right)+A_{2n+1}\left(\frac{8}{q}\right)+\ldots+A_{2n+1}\left(\frac{q-1}{q}\right)\right\}$$
$$=(-1)^n\tfrac{1}{3}I_n.$$

In the group corresponding to § 188, the sign remains the same.

§ 191. The A'-formulæ which are derivable from § 185 are exactly similar to the A-formulæ derived from § 186, the only differences being that q is of another form, and that the sign of the right-hand side is changed.

The first group therefore is

$$\text{(i)}\quad q^{2n}\left\{A'_{2n+1}\left(\frac{1}{q}\right)-A'_{2n+1}\left(\frac{3}{q}\right)+A'_{2n+1}\left(\frac{5}{q}\right)-\ldots-A'_{2n+1}\left(\frac{q-1}{q}\right)\right\}$$
$$=(-1)^n\tfrac{1}{2}E_n,$$

$$\text{(ii)}\quad q^{2n}\left\{A'_{2n+1}\left(\frac{1}{q}\right)-A'_{2n+1}\left(\frac{4}{q}\right)+A'_{2n+1}\left(\frac{7}{q}\right)-\ldots-A'_{2n+1}\left(\frac{q-2}{q}\right)\right\}$$
$$=(-1)^n\tfrac{1}{3}H_n,$$

$$\text{(iii)}\quad q^{2n}\left\{A'_{2n+1}\left(\frac{1}{q}\right)-A'_{2n+1}\left(\frac{5}{q}\right)+A'_{2n+1}\left(\frac{9}{q}\right)-\ldots-A'_{2n+1}\left(\frac{q-3}{q}\right)\right\}$$
$$=(-1)^n\tfrac{1}{2}P_n,$$

$$\text{(iv)}\quad q^{2n}\left\{A'_{2n+1}\left(\frac{1}{q}\right)-A'_{2n+1}\left(\frac{7}{q}\right)+A'_{2n+1}\left(\frac{13}{q}\right)-\ldots-A'_{2n+1}\left(\frac{q-5}{q}\right)\right\}$$
$$=(-1)^n\frac{3^{2n}+1}{4}E_n.$$

In (i) $q \equiv 2$, mod. 4; in (ii) $q \equiv 3$, mod. 6; in (iii) $q \equiv 4$, mod. 8; in (iv) $q \equiv 6$, mod. 12.

§ 192. The group corresponding to § 189 is

$$\text{(i)}\quad q^{2n-1}\left\{A'_{2n}\left(\frac{1}{q}\right)-A'_{2n}\left(\frac{5}{q}\right)+A'_{2n}\left(\frac{9}{q}\right)-\ldots-A'_{2n}\left(\frac{q-3}{q}\right)\right\}$$
$$=(-1)^n\tfrac{1}{2}Q_n,$$

$$\text{(ii)}\quad q^{2n-1}\left\{A'_{2n}\left(\frac{1}{q}\right)-A'_{2n}\left(\frac{7}{q}\right)+A'_{2n}\left(\frac{13}{q}\right)-\ldots-A'_{2n}\left(\frac{q-5}{q}\right)\right\}$$
$$=(-1)^nT_n.$$

In (i) $q \equiv 4$, mod. 8; in (ii) $q \equiv 6$, mod. 12.

§ 193. In the group similar to that in § 191, but having the argument of the first term $\frac{i-1}{q}$, the sign of the right-hand side is the same; but in the group similar to § 192 this sign is changed.

All the formulæ in §§ 174–182 are contained in the preceding sections, or may be derived from them by simple addition or subtraction.

The values of $\lambda_n(q, p)$, &c., *as* n^{th} *derivatives*, §§ 194–201.

§ 194. I proceed now to the consideration of the general functions $\lambda_n(q, p)$, &c. By equating the values of

$$\Sigma_{s=-\infty}^{s=\infty} \frac{1}{(p+2sq)^{2n+1}}$$

derived from §§ 150 and 153, we find that

$$\left(\frac{d}{dp}\right)^{2n} \frac{\pi}{2q} \cot\frac{p\pi}{2q} = (-1)^n \pi^{2n+1} \frac{\lambda'_{2n+1}(q, p)}{q}$$

$$\text{or } = (-1)^{n-1} \pi^{2n+1} \frac{\lambda_{2n+1}(q, p)}{q},$$

according as p is uneven or even.

Now the left-hand side

$$= \left(\frac{d}{dp}\right)^{2n+1} \log \sin\frac{p\pi}{2q},$$

and we may conveniently express the equation as follows: if $x = \frac{p\pi}{2q}$, p and q being any positive integers, then

$$\left(\frac{d}{dx}\right)^{2n+1} \log \sin x = (-1)^n \; 2^{2n+1} q^{2n} \lambda'_{2n+1}(q, p)$$

$$\text{or } (-1)^{n-1} 2^{2n+1} q^{2n} \lambda_{2n+1}(q, p),$$

according as p is uneven or even.

§ 195. For example, let $n=0$ and suppose p uneven. The theorem then gives

$$\cot\frac{p\pi}{2q} = 2\lambda'_1(q, p) = 2\left\{A'_1\left(\frac{1}{q}\right)\sin\frac{p\pi}{q} + A'_1\left(\frac{2}{q}\right)\sin\frac{2p\pi}{q} + \dots \right.$$

$$\left. + A'_1\left(\frac{q-1}{q}\right)\sin\frac{(q-1)p\pi}{q}\right\},$$

which is easily verified, since $A_1(x) = \frac{1}{2}$ for all values of x.

§ 196. As a second example, let $n=1$ and suppose p uneven. The theorem then gives

$$\left(\frac{d}{dx}\right)^3 \log\sin x = -2^3 q^3 \lambda'_3(q, p)$$

$$= -8q^3\left\{A'_3\left(\frac{1}{q}\right)\sin\frac{p\pi}{q} + A'_3\left(\frac{2}{q}\right)\sin\frac{2p\pi}{q} + \dots \right.$$

$$\left. + A'_3\left(\frac{q-1}{q}\right)\sin\frac{(q-1)p\pi}{q}\right\}.$$

Now $A'_3(x) = \frac{1}{2}(x^2 - x)$, and therefore this expression

$$= 4q\left\{\sin\frac{p\pi}{q} + 2\sin\frac{2p\pi}{q} + \dots + (q-1)\sin\frac{(q-1)p\pi}{q}\right\},$$

$$- \;4\left\{\sin\frac{p\pi}{q} + 2^2\sin\frac{2p\pi}{q} + \dots + (q-1)^2\sin\frac{(q-1)p\pi}{q}\right\}.$$

Putting ω for $\frac{p\pi}{q}$, it can be shown that, p being uneven,

$$\sin\omega + 2\sin 2\omega + \dots + (q-1)\sin(q-1)\omega = \frac{q\sin\omega}{2(1-\cos\omega)},$$

and

$$\sin\omega + 2^2\sin 2\omega + \dots + (q-1)^2\sin(q-1)\omega = \frac{(q^2 - 2 - q^2\cos\omega)\sin\omega}{2(1-\cos\omega)^2},$$

so that the expression in question

$$= \frac{4\sin\omega}{(1-\cos\omega)^2} = \frac{2\cos\frac{1}{2}\omega}{\sin^3\frac{1}{2}\omega},$$

which is evidently equal to $\left(\frac{d}{dx}\right)^3 \log\sin x$ when x is put equal to $\frac{1}{2}\omega$.

§ 197. As another example, suppose p even and let $n=0$. Since $A_1(x) = x - \frac{1}{2}$ (§ 67), the theorem becomes

$$\cot\frac{p\pi}{2q} = \sin\frac{p\pi}{q} + \sin\frac{2p\pi}{q} + \dots + \sin\frac{(q-1)p\pi}{q}$$

$$-\frac{2}{q}\left\{\sin\frac{p\pi}{q} + 2\sin\frac{2p\pi}{q} + \dots + (q-1)\sin\frac{(q-1)p\pi}{q}\right\}.$$

The first line on the right-hand side is zero, and the second

$$= -\frac{2}{q} \times \frac{-q\sin\omega}{2(1-\cos\omega)} = \cot\tfrac{1}{2}\omega, \text{ where } \omega = \frac{p\pi}{q}.$$

§ 198. In the same manner we may obtain the formula: if $x=\frac{p\pi}{q}$, p and q being any positive integers, then

$$\left(\frac{d}{dx}\right)^{2n}\log\sin x=(-1)^{n-1}2^{2n}q^{2n-1}\mu'_{2n}(q,p)$$
$$\text{or } (-1)^{n}2^{2n}q^{2n-1}\mu_{2n}(q,p),$$

according as p is uneven or even.

For example, putting $n=1$ and supposing p uneven,

$$\left(\frac{d}{dx}\right)^{2}\log\sin x=4q\mu'_{2}(q,p),$$

that is, since $A'_2(x)=\frac{1}{2}x-\frac{1}{4}$ (§ 66),

$$-\operatorname{cosec}^2\frac{p\pi}{2q}=2\left\{\cos\frac{p\pi}{q}+2\cos\frac{2p\pi}{q}+\ldots+q\cos\frac{qp\pi}{q}\right\}$$
$$-q\left\{\cos\frac{p\pi}{q}+\cos\frac{2p\pi}{q}+\ldots+\cos\frac{qp\pi}{q}\right\},$$

which is easily seen to be true, since, p being uneven, if $\omega=\frac{p\pi}{q}$, then $\cos\omega+\cos 2\omega+\ldots+\cos q\omega=-1$,

and $\cos\omega+2\cos 2\omega+\ldots+q\cos q\omega=-\dfrac{q+2-q\cos\omega}{2(1-\cos\omega)}$.

§ 199. Since

$$\frac{d}{dp}\log\tan\frac{p\pi}{4q}=\frac{\pi}{2q}\operatorname{cosec}\frac{p\pi}{2q},$$

we may also deduce from §§ 150 and 153 the theorems: if $x=\frac{p\pi}{2q}$, then

$$\left(\frac{d}{dx}\right)^{2n+1}\log\tan\tfrac{1}{2}x=(-1)^{n}2^{2n+1}q^{2n}\xi'_{2n+1}(q,p)$$
$$\text{or } (-1)^{n-1}2^{2n+1}q^{2n}\xi_{2n+1}(q,p),$$

and
$$\left(\frac{d}{dx}\right)^{2n}\log\tan\tfrac{1}{2}x=(-1)^{n}2^{2n}q^{2n-1}\zeta'_{2n}(q,p)$$
$$\text{or } (-1)^{n-1}2^{2n}q^{2n-1}\zeta'_{2n}(q,p),$$

the first or second values being taken according as p is uneven or even.

§ 200. We may conveniently express the four results as follows, the arguments (q, p) being omitted after the functions λ', λ, ... for the sake of brevity.

If $x = \dfrac{p\pi}{2q}$, p and q being positive integers, then

$$\text{(i)} \quad \left(\frac{d}{dx}\right)^{2n} \cot x = (-1)^n 2^{2n+1} q^{2n} \lambda'_{2n+1} \text{ or } (-1)^{n-1} 2^{2n+1} q^{2n} \lambda_{2n+1},$$

$$\text{(ii)} \quad \left(\frac{d}{dx}\right)^{2n-1} \cot x = (-1)^{n-1} 2^{2n} q^{2n-1} \mu'_{2n} \text{ or } (-1)^n 2^{2n} q^{2n-1} \mu_{2n},$$

$$\text{(iii)} \quad \left(\frac{d}{dx}\right)^{2n} \frac{1}{\sin x} = (-1)^n 2^{2n+1} q^{2n} \xi'_{2n+1} \quad \text{or } (-1)^{n-1} 2^{2n+1} q^{2n} \xi_{2n+1},$$

$$\text{(iv)} \quad \left(\frac{d}{dx}\right)^{2n-1} \frac{1}{\sin x} = (-1)^n 2^{2n} q^{2n-1} \zeta'_{2n} \quad \text{or } (-1)^{n-1} 2^{2n} q^{2n} \zeta_{2n},$$

the first or second value being taken according as p is uneven or even.

§ 201. The preceding formulæ afford the values of

$$\lambda'_{2n+1}(q, p), \ldots$$

very readily for small values of n. They are, however, chiefly interesting from the opposite point of view as affording the values of the n^{th} derivatives of $\cot x$ or $\operatorname{cosec} x$ for the case in which x is a rational fraction of π, these values being expressed by means of the Bernoullian function.

Recurring formulæ for the calculation of λ_n, ..., §§ 202–209.

§ 202. The values of λ_n, ... for higher values of n may probably be calculated more readily by means of the recurring formulæ derived from the expansions in § 118.

Putting $\omega = \dfrac{p\pi}{q}$ and $\alpha = 1 - \cos\omega$ and supposing p even, we have

$$\frac{\sin\omega}{\alpha + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \&c.} = -2\left\{\lambda_1 + \lambda_3 \frac{(qx)^2}{2!} + \&c.\right\},$$

the arguments (q, p) being omitted after the λ's, since it is always the same.

Multiplying up and equating coefficients, we find

$$aq^{2n}\lambda_{2n+1} + (2n)_2\, q^{2n-2}\lambda_{2n-1} + (2n)_4\, q^{2n-4}\lambda_{2n-3} + \ldots + \lambda_1 = 0,$$

where
$$\lambda_1 = -\frac{\sin\omega}{2(1-\cos\omega)} = -\tfrac{1}{2}\cot\tfrac{1}{2}\omega.$$

Putting $n=2, 3, \ldots,$

$$q^2\lambda_3 = -\frac{1}{a}\lambda_1,$$

$$q^4\lambda_5 = \left(-\frac{1}{a} + \frac{6}{a^2}\right)\lambda_1,$$

$$q^6\lambda_7 = \left(-\frac{1}{a} + \frac{30}{a^2} - \frac{90}{a^3}\right)\lambda_1,$$

$$q^8\lambda_9 = \left(-\frac{1}{a} + \frac{126}{a^2} - \frac{1260}{a^3} + \frac{2520}{a^4}\right)\lambda_1, \quad \text{\&c.}$$

Thus $q^{2n}\lambda_{2n+1}$ is expressed as a series of n terms of powers of $\dfrac{1}{1-\cos\omega}$ multiplied by $\sin\omega$, *e.g.*

$$q^2\lambda_3 = \tfrac{1}{2}\frac{\sin\omega}{(1-\cos\omega)^2},$$

$$q^4\lambda_5 = \tfrac{1}{2}\left\{\frac{1}{(1-\cos\omega)^2} - \frac{6}{(1-\cos\omega)^3}\right\}\sin\omega, \quad \text{\&c.}$$

The values of λ'_{2n+1}, p being uneven, differ only in sign from the corresponding values of λ_{2n+1}, p being even. This is evident, by inspection, from the expansion-formula, or by noticing that the λ''s are connected by the same relation as the λ's, and that $\lambda_1' = \dfrac{\sin\omega}{2(1-\cos\omega)}$.

§ 203. Treating the ξ-expansion in the same way, we have, if p be even,

$$aq^{2n}\xi_{2n+1} + (2n)_2\, q^{2n-2}\xi_{2n-1} + (2n)_4\, q^{2n-4}\xi_{2n-3} + \ldots + \xi_1 = -\frac{1}{2^{2n}}\sin\tfrac{1}{2}\omega,$$

with
$$a\xi_1 = -\sin\tfrac{1}{2}\omega.$$

Thus, putting $n=2, 3, 4, \ldots$, we find

$$\xi_1 = -\frac{1}{a}\sin\tfrac{1}{2}\omega,$$

$$q^2\xi_3 = \left(-\frac{1}{4a}+\frac{1}{a^2}\right)\sin\tfrac{1}{2}\omega,$$

$$q^4\xi_5 = \left(-\frac{1}{16a}+\frac{5}{2a^2}-\frac{6}{a^3}\right)\sin\tfrac{1}{2}\omega,$$

$$q^6\xi_7 = \left(-\frac{1}{64a}+\frac{91}{16a^2}-\frac{105}{2a^3}+\frac{90}{a^4}\right)\sin\tfrac{1}{2}\omega, \ \&c.$$

The values of ξ'_{2n+1}, p being uneven, differ only in sign from those of ξ_{2n+1}.

§ 204. The relation connecting the μ's, p being even, is

$$aq^{2n-1}\mu_{2n} + (2n-1)_2\,q^{2n-3}\mu_{2n-2} + \ldots + (2n-1)_1\,q\mu_2 = \tfrac{1}{2},$$

with $$aq\mu_2 = \tfrac{1}{2}.$$

Thus $$q\mu_2 = \frac{1}{2a},$$

$$q^3\mu_4 = \frac{1}{2a} - \frac{3}{2a^2},$$

$$q^5\mu_6 = \frac{1}{2a} - \frac{15}{2a^2} + \frac{15}{a^3},$$

$$q^7\mu_8 = \frac{1}{2a} - \frac{63}{2a^2} + \frac{210}{a^3} - \frac{315}{a^4}, \ \&c.$$

The values of μ'_{2n}, p being uneven, differ only by a change of sign from those of μ_{2n}.

§ 205. The relation between the ζ's, p being even, is

$$aq^{2n-1}\zeta_{2n} + (2n-1)_2\,q^{2n-3}\zeta_{2n-2} + \ldots + (2n-1)_1\,q\zeta_2 = \frac{\cos\frac{1}{2}\omega}{2^{2n-1}},$$

with

$$aq\zeta_2 = \tfrac{1}{2}\cos\tfrac{1}{2}\omega.$$

Thus $q\zeta_2 = \frac{1}{2\alpha}\cos\frac{1}{2}\omega$,

$$q^3\zeta_4 = \left(\frac{1}{8\alpha} - \frac{3}{2\alpha^2}\right)\cos\tfrac{1}{2}\omega,$$

$$q^5\zeta_6 = \left(\frac{1}{32\alpha} - \frac{15}{4\alpha^2} + \frac{15}{\alpha^3}\right)\cos\tfrac{1}{2}\omega,$$

$$q^7\zeta_8 = \left(\frac{1}{128\alpha} - \frac{273}{32\alpha^2} + \frac{525}{4\alpha^3} - \frac{315}{\alpha^4}\right)\cos\tfrac{1}{2}\omega, \text{ \&c.}$$

The values of ζ'_{2n}, p being uneven, differ only by a change of sign from those of ζ_{2n}.

§ 206. The preceding values of λ_{2n+1}, ... may be conveniently verified by putting $\alpha = 1$ $\left(\text{corresponding to } \cos\omega = 0, \sin\omega = 1, \sin\frac{1}{2}\omega = \cos\frac{1}{2}\omega = \frac{1}{\sqrt{2}}\right)$. When $\alpha = 1$ the coefficients of λ_1 in the values of $q^2\lambda_3$, $q^4\lambda_5$, ... (§ 202) become $-E_1$, E_2, $-E_3$, ...; the coefficients of $\sin\frac{1}{2}\omega$ in the values of ξ_1, $q^2\xi_3$, $q^4\xi_5$, ... (§ 203) become $-P_0$, $\frac{P_1}{2^2}$, $-\frac{P_2}{2^4}$, ... (§ 32); the coefficients of $\cos\frac{1}{2}\omega$ in the values of $q\zeta_2$, $q^3\zeta_4$, $q^5\zeta_6$, ... (§ 205) become $\frac{Q_1}{2}$, $-\frac{Q_2}{2^3}$, $\frac{Q_3}{2^5}$, ... (§ 32): and the values of $q\mu^2$, $q^3\mu_4$, $q^5\mu_6$, ... (§ 204) are

$$2(2^2-1)\frac{B_1}{2},\ -2^3(2^4-1)\frac{B_2}{4},\ 2^5(2^6-1)\frac{B_3}{6},\ \ldots.$$

Other verifications of the same kind are afforded by putting $\alpha = \frac{1}{2}$ and $\alpha = \frac{3}{2}$, corresponding to $\cos\omega = \frac{1}{2}$ and $\cos\omega = -\frac{1}{2}$.

§ 207. We may also obtain recurring relations connecting the λ's, ... in which powers of $1 - \cos 2\omega$ instead of $1 - \cos\omega$ are involved. For multiplying the numerator and denominator in the λ-expansion by $\cosh x + \cos\omega$, and supposing p even, the formula becomes

$$\frac{\sin\omega(\cosh x + \cos\omega)}{\cosh 2x - \cos 2\omega} = -\Sigma_0^\infty \lambda_{2n+1}\frac{(qx)^{2n}}{(2n)!};$$

whence, putting $\beta = 1 - \cos 2\omega$, we find

$$\beta q^{2n}\lambda_{2n+1} + (2n)_2 2^2 q^{2n-2}\lambda_{2n-1} + (2n)_4 2^4 q^{2n-4}\lambda_{2n-3} + \ldots + 2^{2n}\lambda_1 = -\sin\omega,$$

with $$\beta\lambda_1 = -\sin\omega\,(1+\cos\omega).$$

Thus $$q^2\lambda_3 = -\frac{\sin\omega}{\beta} - \frac{2^2}{\beta}\lambda_1$$

$$= -\sin\omega\left\{\frac{1}{\beta} - \frac{4\,(1+\cos\omega)}{\beta^2}\right\}, \text{ \&c.}$$

These values are evidently less simple in form than those in § 202.

The corresponding ξ-formula is

$$\beta q^{2n}\xi_{2n+1} + (2n)_2 2^2 q^{2n-2}\xi_{2n-1} + (2n)_4 2^4 q^{2n-4}\xi_{2n-3} + \ldots + 2^{2n}\xi_1$$
$$= -\frac{3^{2n}+1+2\cos\omega}{2^{2n}}\sin\tfrac{1}{2}\omega,$$

with $$\beta\xi_1 = -2\,(1+\cos\omega)\sin\tfrac{1}{2}\omega.$$

§ 208. The corresponding μ- and ζ-relations are:

$$\beta q^{2n-1}\mu_{2n} + (2n-1)_2 2^2 q^{2n-3}\mu_{2n-2} + \ldots + (2n-1)_1 2^{2n-2} q\mu_2 = 2^{2n-2} + \cos\omega,$$

with $$\beta q\mu_2 = 1 + \cos\omega;$$

and

$$\beta q^{2n-1}\zeta_{2n} + (2n-1)_2 2^2 q^{2n-3}\zeta_{2n-2} + \ldots + (2n-1)_1 2^{2n-2} q\zeta_2$$
$$= \frac{3^{2n-1}-1+2\cos\omega}{2^{2n-1}}\cos\tfrac{1}{2}\omega,$$

with $$\beta q\zeta_2 = (1+\cos\omega)\cos\tfrac{1}{2}\omega.$$

§ 209. We may also obtain relations expressing any one of the functions λ_{2n+1}, μ_{2n}, ... in terms of any other. For example, by division from the λ- and ξ-expansions, we have, p being even,

$$\cosh\tfrac{1}{2}x\Sigma_0^\infty\lambda_{2n+1}\frac{(qx)^{2n}}{(2n)!} = \cos\tfrac{1}{2}\omega\Sigma_0^\infty\xi_{2n+1}\frac{(qx)^{2n}}{(2n)!};$$

whence, equating coefficients,

$$q^{2n}\lambda_{2n+1} + (2n)_1\frac{q^{2n-2}\lambda_{2n-1}}{2^2} + \ldots + \frac{\lambda_1}{2^{2n}} = \xi_{2n+1}.\cos\tfrac{1}{2}\omega,$$

and also

$$q^{2n}\xi_{2n+1}-(2n)_2 E_1 \frac{q^{2n-2}\xi_{2n-1}}{2^2}+\dots(-1)^n E_n \frac{\xi_1}{2^{2n}}=\lambda_{2n+1}.\sec\tfrac{1}{2}\omega.$$

To express μ_{2n} in terms of λ's, we have

$$(2n-1)_1 q^{2n-2}\lambda_{2n-1}+(2n-1)_3 q^{2n-4}\lambda_{2n-3}+\dots(2n-1)_3 q^2\lambda_3+\lambda_1$$
$$=-q^{2n-1}\mu_{2n}\sin\omega,$$

and so on.

The functions λ'_{2n+1}, μ'_{2n}, ..., p being uneven, are connected by exactly the same relations.

Formulæ for the calculation of λ_n, ..., §§ 210–211.

§ 210. It may be remarked that the formulæ of § 212 would probably be of use either in the calculation or verification of the numerical values of λ_{2n+1}, ..., and therefore of a table of $A_{2n+1}\left(\frac{1}{q}\right)$, $A_{2n+1}\left(\frac{2}{q}\right)$, ...

We have

$$\cot x=\frac{1}{x}-2\Sigma_1^\infty \frac{S_{2n}}{\pi^{2n}}x^{2n-1},$$

where
$$S_{2n}=1+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+\&c.;$$

and therefore, taking, for example, the formula (i) of § 200 and performing the differentiations, we find, if p be even,

$$(-1)^{n+1}(2q)^{2n}\pi^{2n+1}\lambda_{2n+1}(q,p)=2^{2n}(2n)!\left(\frac{q}{p}\right)^{2n+1}$$
$$-(2n-1)!\,S_{2n+2}\left(\frac{p}{2q}\right)-\frac{(2n+3)!}{3!}S_{2n+4}\left(\frac{p}{2q}\right)^3-\&c.$$

§ 211. The coefficients S_{2n} approach the limit unity as n increases, and for actual calculation it would probably be convenient to express this equation in a form in which S_{2n+2}, S_{2n+4}, ... are replaced by S'_{2n+2}, S'_{2n+4}, ..., where

$$S'_{2n}=\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+\&c.$$

This may be done most easily by starting with the cotangent-series in the form

$$\cot x = \frac{1}{x} + \frac{1}{x-\pi} + \frac{1}{x+\pi} - 2\Sigma_1^\infty \frac{S'_{2n}}{\pi^{2n}} x^{2n-1};$$

whence we obtain the formula

$$(-1)^{n+1}(2q)^{2n}\pi^{2n+1}\lambda_{2n+1}(q, p)$$
$$= 2^{2n}(2n)!\, q^{2n+1}\left\{\frac{1}{p^{2n+1}} - \frac{1}{(2q-p)^{2n+1}} + \frac{1}{(2q+p)^{2n+1}}\right\}$$
$$-(2n+1)!\, S'_{2n+2}\left(\frac{p}{2q}\right) - \frac{(2n+3)!}{3!} S'_{2n+4}\left(\frac{p}{2q}\right)^3 - \&c.$$

Raabe's investigations, §§ 212–214.

§ 212. The subject of the present paper was suggested to me by some of the formulæ contained in Raabe's memoir "Zurückführung einiger Summen und bestimenten Integrale auf die Jacob-Bernoullischen Function."* The integrals there given indicated the existence of more general formulæ, or groups of formulæ, which it seemed desirable to investigate systematically by means of the Bernoullian function. This was the original object of the paper, and it is proper now to give a brief account of Raabe's results. I have refrained from referring to them in the course of the paper as, for several reasons, it seemed more convenient to place all such references together.

213. Raabe's notation for the Bernoullian function (the introduction of which is due to him) is somewhat different from that employed in this paper, viz. his $B''(x)$ and $B'(x)$ are connected with the A-functions by the relations

$$B''(x) = A_{2m+1}(x),$$

$$B'(x) = A_{2m+2}(x) - A_{2m+2}(0) = A_{2m+2}(x) + (-1)^{m-1}\frac{B_{m+1}}{2m+2}.$$

The function A' was not used in Raabe's memoir.

The fundamental integrals (i) and (ii) of § 9 (Vol. XXVI., p. 156), which express $A_{2n+1}(x)$ and $A_{2n}(x)$ are due to Raabe and were given by him on p. 358 of his memoir.

* *Crelle's Journal*, vol. XLII. (1851), pp. 348–376.

By integrating (ii) by parts he obtained the formula

$$\int_0^1 \log(1-2u\cos 2\pi x+u^2)(\log u)^{2m}\frac{du}{u}$$
$$=\frac{(2\pi)^{2m+2}}{2m+1}\left\{(-1)^{m+1}B'(x)-\frac{B_{m+1}}{2m+2}\right\},$$

which is equivalent to (iii)* of § 60 (Vol. XXVI., p. 181).

On p. 352 Raabe gave the equations

$$\int_0^1 B'(x)\cos 2r\pi x\,dx=0,$$
$$\int_0^1 B'(x)\sin 2r\pi x\,dx=\frac{(-1)^{m-1}}{(2\pi)^{2m+1}}\frac{\Gamma(2m+1)}{r^{2m+1}},$$

with corresponding formulæ involving $B'(x)$. These results are equivalent to the A-formulæ in §§ 75 and 76.

He also deduced, as in § 77, the results (p. 353)

$$\int_0^1 \{B'(x)\}^2 dx=\frac{1.2.3.4\ldots 2m}{(2m+1)(2m+2)\ldots(4m+2)}B_{2m+1},$$
$$\int_0^1 \{B'(x)\}^2\,dx=\frac{1.2.3.4\,..(2m+1)}{(2m+2)(2m+3)\ldots(4m+4)}B_{2m+2}+\frac{B^2_{m+1}}{2m+2},$$

which are equivalent to the A-formulæ in § 80.

§ 214. Raabe obtained also the following general formulæ (p. 362):

(i) $$\int_0^1 \frac{u^{\frac{r}{p}-1}-u^{1-\frac{r}{p}}}{1-u^2}(\log u)^{2m}\,du$$
$$=\frac{(-1)^{m+1}}{p}(2p\pi)^{2m+1}\,\Sigma_{k=1}^{k=p}B'\left(\frac{k}{2p}\right)\sin k\frac{r\pi}{p},$$

(ii) $$\int_0^1 \frac{u^{\frac{r}{p}-1}+u^{1-\frac{r}{p}}}{1+u^2}(\log u)^{2m}\,du$$
$$=\frac{(-1)^{m+1}}{p}(2p\pi)^{2m+1}\,\Sigma_{k=1}^{k=p}B'\left(\frac{2k-1}{4p}\right)\sin(2k-1)\frac{r\pi}{2p}.$$

* There is an obvious misprint in this formula; the expression subject to the logarithm should be $1-2e^{-t}\cos 2\pi x+e^{-2t}$.

Transforming these integrals by putting $u=e^{-pt}$ and using the A-notation for the Bernoullian function, these equations become

$$\int_0^\infty \frac{\sinh(p-r)t}{\sinh pt} t^{2m} dt = \frac{(-1)^{m+1}}{p}(2\pi)^{2m+1} \Sigma_{k=1}^{k=p} A_{2m+1}\left(\frac{k}{2p}\right) \sin\frac{kr\pi}{p},$$

$$\int_0^\infty \frac{\cosh(p-r)t}{\cosh pt} t^{2m} dt$$

$$= \frac{(-1)^{m+1}}{p}(2\pi)^{2m+1} \Sigma_{k=1}^{k=p} A_{2m+1}\left(\frac{2k-1}{4p}\right) \sin\frac{(2k-1)r\pi}{2p}.$$

Now $$\Sigma_{k=1}^{k=p} A_{2m+1}\left(\frac{k}{2p}\right) \sin\frac{kr\pi}{p} = \tfrac{1}{2}\lambda_{2m+1}(2p, 2r),$$

and $$\Sigma_{k=1}^{k=p} A_{2m+1}\left(\frac{2k-1}{4p}\right) \sin\frac{(2k-1)r}{2p} = \tfrac{1}{2}\xi_{2m+1}(2p, 2r),$$

so that the values of the two integrals are respectively

$$\frac{(-1)^{m+1}}{p} 2^{2m} \pi^{2m+1} \lambda_{2m+1}(2p, 2r),$$

and $$\frac{(-1)^{m+1}}{p} 2^{2m} \pi^{2m+1} \xi_{2m+1}(2p, 2r).$$

These results agree with those given in § 128 by virtue of the relations

$$2^{2n}\lambda_{2n+1}(2q, 2p) = \lambda_{2n+1}(q, p) \text{ or } -\lambda'_{2n+1}(q, p), \text{ \&c.},$$

which were obtained in § 143.*

It is interesting to notice the form in which Raabe obtained the values of these integrals. His method is quite different from that employed in this paper, and, as he did not use the A'-functions, he could not have expressed his results in the same form as in § 128.

The investigations contained in §§ 110–130 (and therefore also the introduction of the functions λ_n, λ'_n, ...) were suggested by the formulæ (i) and (ii).

---

* Raabe also transformed the integrals in the same manner as in § 130.

§ 215. From the first of these formulæ, Raabe deduced the relation

$$\Sigma_{k=1}^{k=p}(-1)^{k-1}B'\left(\frac{2k-1}{4p}\right)=\frac{1}{p^{2m}}B'(\tfrac{1}{4}),$$

which is the same as (i) of § 178, and therefore belongs to the class of formulæ discussed in §§ 166–193.

§ 216. Raabe specially noticed the three particular cases of the formulæ (i) and (ii), viz. (p. 361)

$$\int_0^1 \frac{(\log u)^{2m}}{1+u^2}\,du=\frac{(-1)^{m+1}}{2}(2\pi)^{2m+1}B'(\tfrac{1}{4}),$$

$$\int_0^1 \frac{(\log u)^{2m}}{1+u+u^2}\,du=\frac{(-1)^{m+1}}{\surd 3}(2\pi)^{2m+1}B'(\tfrac{1}{3}),$$

$$\int_0^1 \frac{(\log u)^{2m}}{1-u+u^2}\,du=\frac{(-1)^{m+1}}{\surd 3}(2\pi)^{2m+1}B'(\tfrac{1}{6}).$$

Substituting for $A_{2n+1}(\frac{1}{4})$, &c., their values from § 55, and putting $u=e^{-t}$, these equations become

$$\int_0^\infty \frac{t^{2m}dt}{e^t+e^{-t}}=\tfrac{1}{2}(\tfrac{1}{2}\pi)^{2m+1}E_m,$$

$$\int_0^\infty \frac{t^{2m}dt}{e^t+e^{-t}+1}=\frac{1}{\surd 3}(\tfrac{2}{3}\pi)^{2m+1}J_m,$$

$$\int_0^\infty \frac{t^{2m}dt}{e^t+e^{-t}-1}=\frac{1}{\surd 3}(\tfrac{1}{3}\pi)^{2m+1}J_m,$$

which are equivalent to (iv) of § 34, and (i) and (iii) of § 36.

The notation $V_n(x)$ and $U_n(x)$, § 217.

§ 217. In the second part of the paper in the *Quarterly Journal** I have used, in place of $A_n(x)$ and $A_n'(x)$, two other functions $V_n(x)$ and $U_n(x)$, which are connected with $A_n(x)$ and $A_n'(x)$ by the simple relations

$$V_n(x)=nA_n(x), \qquad U_n(x)=nA_n'(x).$$

The V- and U-form is better adapted to symbolic treatment, and for several other reasons it is to be preferred as a permanent notation. The A- and A'-notation was convenient for the expression of the results contained in the earlier portion of the

* Vol. XXIX., p. 114.

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Articles for insertion should be sent to Dr. Glaisher, Trinity College, Cambridge, or to Messrs. Metcalfe and Co. Limited, Printing Office, Trinity Street, Cambridge.

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present paper, and it has been retained throughout for the sake of clearness. It is easy to pass almost at sight from the A's and A''s to the V's and U's, and it is evident that many of the results when so expressed assume a more natural form.

CONTENTS.

ERRATA.

Vol. xxvi. p. 169 l. 9 *for* cosh $\frac{1}{2}a$ *read* cosh $\frac{1}{4}a$.

,, p. 171, l. 14, 15, 17 *for*
$\frac{\sinh x}{\cosh 3x}$, $\frac{\sinh x}{\cosh 3x}$, $\frac{\sinh 2x}{\cosh 2x}$ *read* $\frac{\cosh x}{\cosh 2x}$, $\frac{\sinh x}{\cosh 2x}$, $\frac{\sinh 2x}{\cosh 3x}$.

,, p. 175, l. 11 *for* $\frac{\cosh 2\pi t}{\cosh \pi t}$ *read* $\frac{\cosh \pi t}{\cosh 2\pi t}$.

,, p. 179, l. 4 *for* $\frac{B_n}{2n}$ *read* $\frac{B_n}{4n}$.

,, p. 181, l. 15. The quantity after log should be $1-2e^{-t}\cos 2\pi x+e^{-2t}$.

Vol. xxvii. p. 25, l. 6 *for* $A_{2n}(x)\sin 2r\pi x$ *read* $A_{2n}\cos 2r\pi x$.

NEW SOLUTIONS OF SOME OF THE PARTIAL DIFFERENTIAL EQUATIONS OF MATHEMATICAL PHYSICS.

By *Professor A. R. Forsyth.*

THE results in the present paper were originally obtained by the application of a method which I have devised for solving partial differential equations of the second order in more than two variables; and a paper, containing an account of the method, will soon be published. The particular results, as they have admitted of immediate verification and as they have an independent interest, have been abstracted for separate publication; they relate, as will be seen, to the important equations

$$\nabla^2 v = 0,$$

$$\nabla^2 v = -\kappa^2 v,$$

$$\nabla^2 v = c^2 \ddot{v},$$

which occur in mathematical physics.

Two remarks should be added.

First, my aim in each case has been to obtain a solution, and not, by means of the analysis, to construct the most general solution which that analysis permits. Thus, in § 8, with the equation

$$p^2 + q^2 + r^2 = 0,$$

four distinct variables u_1, u_2, u_3, u_4 can be obtained from the equations

$$\left.\begin{aligned} au_1 &= xp(u_1) + yq(u_1) + zr(u_1) \\ au_2 &= -xp(u_2) + yq(u_2) + zr(u_2) \\ au_3 &= xp(u_3) - yq(u_3) + zr(u_3) \\ au_4 &= -xp(u_4) - yq(u_4) + zr(u_4) \end{aligned}\right\} ;$$

corresponding to each of them there is a solution of the type there indicated, and the sum of the four solutions thus given is a more general solution. Further, the sum of any number of these more general solutions, corresponding to different determinations of p, q, r, is also a solution.

Secondly, even this solution of the equation is not the most general solution the existence of which is indicated by theory.

I. *Laplace's equation* $\nabla^2 v = 0$.

1. Let $p_1, p_2, \ldots, p_n$ denote n arbitrary functions of a variable u, subject to the single condition

$$p_1^2 + p_2^2 + \ldots + p_n^2 = 0;$$

and let u be determined as a function of n independent variables $x_1, x_2, \ldots, x_n$ by the equation

$$au = x_1 p_1 + x_2 p_2 + \ldots + x_n p_n,$$

where a is a constant. Then *if v denote any arbitrary function of u, it satisfies the equation*

$$\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2} + \ldots + \frac{\partial^2 v}{\partial x_n^2} = 0.$$

The proof is simple. Denoting

$$a - x_1 p_1' - x_2 p_2' - \ldots - x_n p_n'$$

by Δ, we have

$$\frac{\partial v}{\partial x_s} = \frac{\partial v}{\partial u} \frac{p_s}{\Delta},$$

$$\frac{\partial^2 v}{\partial x_s^2} = \frac{\partial^2 v}{\partial u^2} \frac{p_s^2}{\Delta^2} + \frac{\partial v}{\partial u} \frac{p_s' p_s}{\Delta^2} + \frac{\partial v}{\partial u} \frac{p_s}{\Delta^2} \left\{ p_s' + \frac{p_s}{\Delta} \sum_{r=1}^{n} x_r p_r'' \right\};$$

consequently

$$\sum_{s=1}^{n} \frac{\partial^2 v}{\partial x_s^2} = \frac{1}{\Delta^2} \frac{\partial^2 v}{\partial u^2} \sum_{s=1}^{n} p_s^2 + \frac{2}{\Delta^2} \frac{\partial v}{\partial u} \sum_{s=1}^{n} p_s p_s' + \frac{1}{\Delta^3} \frac{\partial v}{\partial u} \sum_{r=1}^{n} x_r p_r'' \sum_{r=1}^{n} p_s^2.$$

Each term on the right-hand side vanishes; for we have

$$\sum_{s=1}^{n} p_s^2 = 0,$$

and therefore

$$\sum_{s=1}^{n} p_s p_s' = 0;$$

hence

$$\sum_{s=1}^{n} \frac{\partial^2 v}{\partial x_s^2} = 0,$$

the required equation.

And it should be noticed that we have

$$\frac{1}{p_1}\frac{\partial v}{\partial x_1} = \frac{1}{p_2}\frac{\partial v}{\partial x_2} = \ldots = \frac{1}{p_n}\frac{\partial v}{\partial x_n},$$

$$\left(\frac{\partial v}{\partial x_1}\right)^2 + \left(\frac{\partial v}{\partial x_2}\right)^2 + \ldots + \left(\frac{\partial v}{\partial x_n}\right)^2 = 0.$$

2. The simplest case arises when $n=2$. Denoting the variables by x and y, and the arbitrary functions by p and q, we have

$$p^2 + q^2 = 0,$$

say $q = ip$; and then, taking $a=1$, we have

$$\begin{aligned} u &= px + qy \\ &= p(x+iy), \end{aligned}$$

so that u (and consequently any arbitrary function of u) is an arbitrary function of $x+iy$; and this satisfies the equation

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Moreover, we have

$$\frac{1}{p}\frac{\partial v}{\partial x} = \frac{1}{q}\frac{\partial v}{\partial y},$$

or

$$\frac{\partial v}{\partial y} = i\frac{\partial v}{\partial x};$$

and

$$\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 = 0.$$

The quantity v is complex; when we take

$$v = \phi + i\psi,$$

where ϕ and ψ are real functions, and substitute, we have the ordinary equations characteristic of a two-dimensional potential. Both ϕ and ψ satisfy

$$\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} = 0;$$

and when either of them, is given the other can be constructed by quadratures from the equations

$$\frac{\partial\psi}{\partial y} = \frac{\partial\phi}{\partial x},$$

$$\frac{\partial\psi}{\partial x} = -\frac{\partial\phi}{\partial y}.$$

We, in fact, obtain the ordinary theory of the conjugate two-dimensional potentials.

3. The case $n=3$ furnishes a corresponding result, giving rise to conjugate potentials in three dimensions. Denoting the variables by x, y, z and the arbitrary functions by p, q, r, we have the variable u determined by

$$au = xp + yq + zr;$$

and any arbitrary function of u, say v, satisfies the equation

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0.$$

Moreover, we have

$$\frac{1}{p}\frac{\partial v}{\partial x} = \frac{1}{q}\frac{\partial v}{\partial y} = \frac{1}{r}\frac{\partial v}{\partial z},$$

$$\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 = 0.$$

The function v is complex; let it be denoted by $\phi + i\psi$, where ϕ and ψ are real functions of x, y, z. Then both ϕ and ψ satisfy the potential equation

$$\nabla^2\xi = 0;$$

and substituting in the equation

$$\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 = 0,$$

the value $\phi + i\psi$ for v, we have

$$\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2 = \left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2,$$

$$\frac{\partial\phi}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\psi}{\partial y} + \frac{\partial\phi}{\partial z}\frac{\partial\psi}{\partial z} = 0.$$

In other words, the function v determines two conjugate potentials; the forces due to these potentials are equal in magnitude and perpendicular in directions.

This property can be expressed differently as follows. Since ϕ satisfies the equation

$$\nabla^2\phi = 0,$$

it can be taken as a velocity-potential, say for a fluid; so that, if (using for a moment the symbols u, v, w in another sense), u, v, w be the components of the velocity, we have

$$u = -\frac{\partial\phi}{\partial x}, \quad v = -\frac{\partial\phi}{\partial y}, \quad w = -\frac{\partial\phi}{\partial z}.$$

We thus have

$$u\frac{\partial\psi}{\partial x} + v\frac{\partial\psi}{\partial y} + w\frac{\partial\psi}{\partial z} = 0,$$

so that the stream-line at any point lies in that surface $\psi =$ constant which passes through the point; in other words, $\psi =$ constant is a current-sheet. To obtain the equations of the stream-line, the equation of some other surface must be combined with $\psi =$ constant $= \gamma$; one simple plan is to use $\psi = \gamma$ to determine z as a function of x and y, and then, substituting in the equation

$$\frac{dx}{-\dfrac{\partial\phi}{\partial x}} = \frac{dy}{-\dfrac{\partial\phi}{\partial y}},$$

to integrate the modified form. If this be

$$S_1(x, y) = \text{constant} = \delta,$$

S_1 in general would involve γ; substituting for γ, we should have a form

$$S(x, y, z) = \delta;$$

and then the stream-line through a point x_0, y_0, z_0 would be given by

$$\left.\begin{aligned} \psi(x, y, z) &= \psi(x_0, y_0, z_0) \\ S(x, y, z) &= S(x_0, y_0, z_0) \end{aligned}\right\}.$$

4. Examples can easily be constructed. The relation

$$p^2 + q^2 + r^2 = 0$$

is satisfied by

$$\left.\begin{aligned} p &= u^2 - 1 \\ q &= i(u^2 + 1) \\ r &= 2u \end{aligned}\right\};$$

so that, writing $2c$ in place of a, we have

$$2cu = (u^2 - 1)\,x + i\,(u^2 + 1)\,y + 2uz,$$

and therefore

$$u = \frac{c - z \pm \{(c - z)^2 + x^2 + y^2\}^{\frac{1}{2}}}{x + iy}.$$

Then v is any arbitrary function of u; if, in particular, it be taken equal to u, we have

$$\begin{aligned} \phi + i\psi &= u \\ &= \frac{c - z \pm \{(c - z)^2 + x^2 + y^2\}^{\frac{1}{2}}}{x + iy}; \end{aligned}$$

and therefore

$$\phi = \frac{c - z \pm \{(c - z)^2 + x^2 + y^2\}^{\frac{1}{2}}}{x^2 + y^2}\,x,$$

$$\psi = -\frac{c - z \pm \{(c - z)^2 + x^2 + y^2\}^{\frac{1}{2}}}{x^2 + y^2}\,y,$$

are conjugate potentials of the kind indicated.

If the hydrodynamical illustration be developed, then ϕ is the velocity-potential: and the stream-line is given by

$$\frac{dx}{-\dfrac{\partial\phi}{\partial x}} = \frac{dy}{-\dfrac{\partial\phi}{\partial y}} = \frac{dz}{-\dfrac{\partial\phi}{\partial z}}.$$

It is easy to verify that one integral of this system is given by $\psi =$ constant $= -\alpha$ say; so that, writing

$$(c - z)^2 + x^2 + y^2 = r^2,$$

$$x^2 + y^2 = \rho^2,$$

we find

$$c - z = \frac{a^2\rho^2 - y^2}{2ay},$$

and

$$r = \frac{a^2\rho^2 + y^2}{2ay}.$$

Then

$$\frac{\partial\phi}{\partial x} = \frac{c - z + r}{\rho^2}\frac{y^2 - x^2}{\rho^2} + \frac{x^2}{r\rho^2}$$

$$= \frac{a}{y(y^2 + a^2\rho^2)}\{y^2(1 + a^2) - a^2x^2\},$$

after reduction; and

$$\frac{\partial\phi}{\partial y} = -\frac{c - z + r}{\rho^2}\frac{2xy}{\rho^2} + \frac{xy}{r\rho^2}$$

$$= -\frac{2a^2x}{y^2 + a^2\rho^2},$$

after reduction. Substituting in the first of the equations for the stream-line, we have

$$2x\frac{dx}{dy}a^2 - a^2\frac{x^2}{y} + y(1 + a^2) = 0,$$

whence

$$a^2x^2 + y^2(1 + a^2) = by,$$

where b is arbitrary. Hence the stream-lines for the velocity-potential

$$\frac{c - z \pm \{(c - z)^2 + x^2 + y^2\}^{\frac{1}{2}}}{x^2 + y^2}x$$

are given by

$$\left.\begin{aligned} \frac{c - z \pm \{(c - z)^2 + x^2 + y^2\}^{\frac{1}{2}}}{x^2 + y^2} &= \frac{a}{y} \\ \text{and} \qquad a^2x^2 + y^2(1 + a^2) &= by \end{aligned}\right\},$$

The former equation, which contains the potential conjugate to ϕ, can be replaced by a rational equivalent as above; and the stream-lines are given by

$$\left.\begin{aligned} a^2x^2 + y^2(a^2 - 1) &= 2ay(c - z) \\ a^2x^2 + y^2(a^2 + 1) &= by \end{aligned}\right\},$$

where a and b are the parameters of the stream-line. (In the particular instance, the stream lines are plane curves).

5. The functions ϕ and ψ, determined by the foregoing method, are not solely or completely determined as solutions of the potential-equation in free space. The quantity u is complex, and therefore, also p, q, r are complex. Let

$$p = (\alpha + i\alpha')\, r, \quad q = (\beta + i\beta')\, r,$$

where α, β, α', β' are real quantities; then from the relation $p^2 + q^2 + r^2 = 0$, we have

$$\left.\begin{array}{l} \alpha^2 + \beta^2 + 1 = \alpha'^2 + \beta'^2 \\ \alpha\alpha' + \beta\beta' = 0 \end{array}\right\}.$$

Now we have

$$\frac{1}{p}\frac{\partial v}{\partial x} = \frac{1}{q}\frac{\partial v}{\partial y} = \frac{1}{r}\frac{\partial v}{\partial z},$$

so that

$$\frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = (\alpha + i\alpha')\left(\frac{\partial \phi}{\partial z} + i\frac{\partial \psi}{\partial z}\right),$$

$$\frac{\partial \phi}{\partial y} + i\frac{\partial \psi}{\partial y} = (\beta + i\beta')\left(\frac{\partial \phi}{\partial z} + i\frac{\partial \psi}{\partial z}\right);$$

and therefore

$$\left.\begin{array}{ll} \dfrac{\partial \phi}{\partial x} = \alpha\dfrac{\partial \phi}{\partial z} - \alpha'\dfrac{\partial \psi}{\partial z}, & \dfrac{\partial \psi}{\partial x} = \alpha'\dfrac{\partial \phi}{\partial z} + \alpha\dfrac{\partial \psi}{\partial z} \\ \dfrac{\partial \phi}{\partial y} = \beta\dfrac{\partial \phi}{\partial z} - \beta'\dfrac{\partial \psi}{\partial z}, & \dfrac{\partial \psi}{\partial y} = \beta\dfrac{\partial \phi}{\partial z} + \beta\dfrac{\partial \psi}{\partial z} \end{array}\right\}.$$

Among these four equations, we can eliminate the three derivatives of ϕ; the result, by using the relation $\alpha\alpha' + \beta\beta' = 0$, can be expressed in the form

$$\alpha\frac{\partial \psi}{\partial x} + \beta\frac{\partial \psi}{\partial y} = (\alpha^2 + \beta^2)\frac{\partial \psi}{\partial z}.$$

Or we can eliminate the three derivatives of ψ; the result can be expressed in the form

$$\alpha\frac{\partial \phi}{\partial x} + \beta\frac{\partial \phi}{\partial y} = (\alpha^2 + \beta^2)\frac{\partial \phi}{\partial z}.$$

Hence it appears that the conjugate potentials ϕ and ψ satisfy, each of them, the equation

$$\alpha\frac{\partial\xi}{\partial x}+\beta\frac{\partial\xi}{\partial y}=(\alpha^2+\beta^2)\frac{\partial\xi}{\partial z};$$

but it must be borne in mind that, in this equation, the quantities α and β (which arise through p and q) are not entirely independent of ϕ and ψ.

Moreover, if either ϕ or ψ be given, the other can be constructed by quadratures; but for effective integration, the quantities α and β must be considered known, and therefore the quadratures (which are given by the preceding equation) are not simple quadratures when the function alone is given and not its source.

6. Another solution, generally distinct from that in § 1, is given by

$$v=\frac{G(u)}{\Delta}=\frac{G}{\Delta}$$

say, where G is an arbitrary function of u. The verification again is simple. We have

$$\frac{\partial v}{\partial x_s}=\frac{G'}{\Delta}\frac{p_s}{\Delta}+\frac{G}{\Delta^2}\left\{p_s'+\frac{p_s}{\Delta}\sum_{r=1}^{n}x_rp_r''\right\},$$

and therefore

$$\begin{aligned}\frac{\partial^2 v}{\partial x_s^2}=&\frac{G''p_s+2G'p_s'+Gp_s''}{\Delta^2}\frac{p_s}{\Delta}\\&+2\frac{G'p_s+Gp_s'}{\Delta^3}\left\{p_s'+\frac{p_s}{\Delta}\sum_{r=1}^{n}x_rp_r''\right\}\\&+\frac{G'p_s+Gp_s'}{\Delta^3}\frac{p_s}{\Delta}\sum_{r=1}^{n}x_rp_r''\\&+\frac{3Gp_s}{\Delta^4}\left\{p_s'+\frac{p_s}{\Delta}\sum_{m=1}^{n}x_mp_m''\right\}\sum_{r=1}^{n}x_rp_r''\\&+\frac{Gp_s}{\Delta^3}\left\{p_s''+\frac{p_s}{\Delta}\sum_{r=1}^{n}x_rp_r'''\right\}.\end{aligned}$$

Now we have

$$\sum_{s=1}^{n} p_s^2 \quad = 0,$$

and therefore

$$\sum_{s=1}^{n} p_s p_s' = 0,$$

and

$$\sum_{s=1}^{n} p_s p_s'' + \sum_{s=1}^{n} p_s'^2 = 0.$$

When we form the sum $\sum_{s=1}^{n} \frac{\partial^2 v}{\partial x_s^2}$ and take account of these relations, it appears that the sum vanishes: in other words, v satisfies the equation

$$\sum_{s=1}^{n} \frac{\partial^2 v}{\partial x_s^2} = 0.$$

Moreover, it at once follows, from the expressions for the first derivatives of v, that

$$\left(\frac{\partial v}{\partial x_1}\right)^2 + \left(\frac{\partial v}{\partial x_2}\right)^2 + \ldots + \left(\frac{\partial v}{\partial x_n}\right)^2$$

can vanish only when $\sum_{s=1}^{n} p_s'^2 = 0$: a condition, which is not in general satisfied.

7. The simplest case arises, as before, when $n = 2$; but the new solution is not, in fact, different from the former. For

$$\Delta = 1 - xp' - yq',$$

with the notation of § 2, or

$$\Delta = 1 - p'(x + iy).$$

Moreover, u is a function of $x + iy$, as therefore, also is p'; consequently $\frac{G}{\Delta} =$ arbitrary function of $x + iy$, which is the former solution.

8. Taking next the case $n = 3$, and adopting the notation of § 3, we have

$$au = xp + yq + zr,$$

$$\Delta = a - xp' - yq' - zr';$$

and then

$$v_1 = \frac{G_1(u)}{a - xp' - yq' - zr'},$$

satisfies Laplace's equation

$$\nabla^2 v_1 = 0,$$

G_1 denoting any arbitrary function. Except in the very special when $p : q : r$ are constant ratios, this solution is distinct from

$$v = \phi(u).$$

The two solutions, it is easy to see, can be combined; and they lead to the theorem*:

If p, q, r be any arbitrary functions of u, subject to the single condition

$$p^2 + q^2 + r^2 = 0;$$

and if u be defined as a function of x, y, z by the equation

$$au = xp + yq + zr,$$

where a is a constant, then

$$v = \frac{\psi(u)}{a - xp' - yq' - zr'} + \phi(u),$$

where ϕ and ψ denote arbitrary functions, is a solution of Laplace's equation

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0.$$

Moreover, we have

$$\begin{aligned}\frac{\partial v}{\partial x} = \frac{\psi'(u)}{(a - xp' - yq' - zr')^2}\, p \\ + \frac{\psi'(u)}{(a - xp' - yq' - zr')^2} \left\{p' + \frac{p\,(xp'' + yq'' + zr'')}{a - xp' - yq' - zr'}\right\} \\ + \frac{\phi'(u)}{a - xp' - yq' - zr'}\, p,\end{aligned}$$

* See also § 15 of this paper.

with similar values for $\frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial z}$; consequently,

$$\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2$$

vanishes only if $p'^2 + q'^2 + r'^2 = 0$. This condition requires that p, q, r are constant ratios; and the solution then is not distinct from that in § 1.

As before, v is a complex quantity; denoting it by $U + iV$, where U and V are real, then U and V are solutions of Laplace's equation.

Also U can be taken as a velocity-potential for fluid motion in three dimensions; $V =$ constant is not now a current-sheet. The equations necessary to determine the stream-line can be determined by integrating the equations

$$\frac{dx}{-\frac{\partial U}{\partial x}} = \frac{dy}{-\frac{\partial U}{\partial y}} = \frac{dz}{-\frac{\partial U}{\partial z}}.$$

II. *The equation* $\nabla^2 v + \kappa^2 v = 0$.

9. The general equation for the conduction of heat in a homogeneous body, and the general equation for the propaga- of sound in a homogeneous medium, can be made (as to their solution) to depend upon the solution of the equation

$$\nabla v^2 + \kappa^2 v = 0,$$

where κ denotes a constant.

With the notation of § 1, we take

$$au = x_1 p_1 + x_2 p_2 + \ldots + x_n p_n,$$

where $p_1, p_2, \ldots, p_n$ are arbitrary functions of u, subject to the single condition

$$\sum_{s=1}^{n} p_s^2 = 0.$$

From the latter, it follows that

$$\sum_{s=1}^{n} p_s p_s' = 0,$$

and
$$\sum_{s=1}^{n} p_s p_s'' = -\sum_{s=1}^{n} p_s'^2 = -\theta^2,$$

where, in general, θ is a complex function of u. Also, let

$$\eta = x_1 p_1' + x_2 p_2' + \ldots + x_n p_n',$$
$$\rho = x_1 p_1'' + x_2 p_2'' + \ldots + x_n p_n'',$$
$$\sigma = x_1 p_1''' + x_2 p_2''' + \ldots + x_n p_n''',$$

so that

$$\frac{\partial u}{\partial x_s} = \frac{p_s}{a - \eta},$$
$$\frac{\partial \eta}{\partial x_s} = p_s' + \frac{p_s}{a - \eta} \rho,$$
$$\frac{\partial \rho}{\partial x_s} = p'' + \frac{p_s}{a - \eta} \sigma.$$

Other considerations, that are not dealt with here, suggest a form of solution; but it will be sufficient to find, on the analogy of the solution of $\nabla^2 u = 0$ given in § 8, what solutions (if any) of the form

$$v = f(u, \eta)$$

are possessed by the equation

$$\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2} + \ldots + \frac{\partial^2 v}{\partial x_n^2} + \kappa^2 v = 0,$$

κ being a constant. We have

$$\frac{\partial v}{\partial x_s} = \frac{\partial f}{\partial u} \frac{p_s}{a - \eta} + \frac{\partial f}{\partial \eta} \left(p_s' + \frac{p_s}{a - \eta} \rho\right),$$

and therefore

$$\begin{aligned}\frac{\partial^2 v}{\partial x_s^2} = \frac{\partial^2 f}{\partial u^2} \left(\frac{p_s}{a - \eta}\right)^2 &+ 2 \frac{\partial^2 f}{\partial u \, \partial \eta} \frac{p_s}{a - \eta} \left(p_s' + \frac{p_s}{a - \eta} \rho\right) \\ &+ \frac{\partial^2 f}{\partial \eta^2} \left(p_s' + \frac{p_s}{a - \eta} \rho\right)^2 \\ &+ \frac{\partial f}{\partial u} \left[\frac{p_s'}{a - \eta} \frac{p_s}{a - \eta} + \frac{p_s}{(a - \eta)^2} \left\{p_s' + \frac{p_s}{a - \eta} \rho\right\}\right] \\ &+ \frac{\partial f}{\partial \eta} \left[p_s'' \frac{p_s}{a - \eta} + \frac{p_s}{a - \eta} \left(p_s'' + \frac{p_s}{a - \eta} \sigma\right)\right. \\ &\qquad \left. + \frac{p_s' \rho}{a - \eta} \frac{p_s}{a - \eta} + \frac{p_s \rho}{(a - \eta)^2} \left\{p_s' + \frac{p_s}{a - \eta} \rho\right\}\right].\end{aligned}$$

Hence, adding and taking account of the values of $\sum\limits_{s=1}^{n} p_s^2$, $\sum\limits_{s=1}^{n} p_s p_s'$, $\sum\limits_{s=1}^{n} p_s p_s''$, $\sum\limits_{s=1}^{n} p_s'^2$, we have

$$\sum_{s=1}^{n} \frac{\partial^2 v}{\partial x_s^2} = \theta^2 \frac{\partial^2 f}{\partial \eta^2} - 2 \frac{\theta^2}{a-\eta} \frac{df}{d\eta};$$

consequently the equation determining f is

$$\theta^2 \frac{\partial^2 f}{\partial \eta^2} - 2 \frac{\theta^2}{a-\eta} \frac{\partial f}{\partial \eta} + \kappa^2 f = 0.$$

This can be written in the form

$$\frac{\partial^2}{\partial \eta^2} \{(a-\eta) f\} + \frac{\kappa^2}{\theta^2} (a-\eta) f = 0;$$

and its most general integral is

$$(a-\eta) f = Ae^{i\frac{\kappa\eta}{\theta}} + Be^{-i\frac{\kappa\eta}{\theta}},$$

where A and B are independent of η: that is, since f involves u, we have A and B as arbitrary functions of u, say $\phi(u)$ and $\psi(u)$ respectively. Hence it appears that a solution of the equation

$$\sum_{s=1}^{n} \frac{\partial^2 v}{\partial x_s^2} + \kappa^2 v = 0$$

is given by

$$v = \frac{\phi(u)}{a-\eta} \exp. \left\{ i\kappa \frac{\sum\limits_{s=1}^{n} x_s p_s'}{\left(\sum\limits_{s=1}^{n} p_s'^2\right)^{\frac{1}{2}}} \right\} + \frac{\psi(u)}{a-\eta} \exp. \left\{ -i\kappa \frac{\sum\limits_{s=1}^{n} x_s p_s'}{\left(\sum\limits_{s=1}^{n} p_s'^2\right)^{\frac{1}{2}}} \right\},$$

where ϕ and ψ are arbitrary functions.

10. The analysis does not apply to the case $n=2$; for we then have

$$p^2 + q^2 = 0,$$

and so

$$\eta = xp' + yq'$$
$$= \frac{p'}{p}(px+qy)$$
$$= u\frac{p'}{p},$$

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whence η cannot be taken as an independent variable, in addition to u.

Taking u', $=x-iy$, and u, $=x+iy$, as the independent variables—substitutions suggested by $p^2+q^2=0$; $u=px+qy$, if otherwise not known for this case—the equation for v becomes

$$\frac{\partial^2 v}{\partial u \partial u'} + \tfrac{1}{4}\kappa^2 v = 0,$$

an instance of Laplace's linear equation in its canonical form. The two invariants are equal to one another and are constant, κ being a constant; the Darboux-sequence of derived invariants is infinite and therefore the solution, containing the arbitrary functions in explicit linear form, is not expressible in finite terms.*

11. Taking next the case $n=3$, we have

$$au = xp + yq + zr,$$
$$0 = p^2 + q^2 + r^2;$$

a solution of the equation

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} + \kappa^2 v = 0$$

is given by

$$v = \frac{\phi(u)}{a - xp' - yq' - zr'} e^{i\kappa \frac{xp'+yq'+zr'}{(p'^2+q'^2+r'^2)^{\frac{1}{2}}}} + \frac{\psi(u)}{a - xp' - yq' - zr'} e^{-i\kappa \frac{xp'+yq'+zr'}{(p'^2+q'^2+r'^2)^{\frac{1}{2}}}},$$

where ϕ and ψ are arbitrary functions.

Thus for the special instance in § 4, when

$$p = u^2 - 1, \quad q = i(u^2+1), \quad r = 2u,$$

we have

$$p'^2 + q'^2 + r'^2 = 4,$$

and

$$\begin{aligned} xp' + yq' + zr' &= 2\{(x+iy)u + z\} \\ &= 2c \pm 2\{(c-z)^2 + x^2 + y^2\}^{\frac{1}{2}}, \end{aligned}$$

* Darboux, *Théorie générale des surfaces* (vol. ii., p. 37, and elsewhere in book iv. of vol. ii.).

and $a=2c$. Hence, on absorbing the exponential constants $e^{i\kappa c}$ and $e^{-i\kappa c}$ into ϕ and ψ respectively, it follows that

$$v=\frac{1}{R}\phi\left(\frac{c-z+R}{x+iy}\right)e^{i\kappa R}+\frac{1}{R}\psi\left(\frac{c-z+R}{x+iy}\right)e^{-i\kappa R},$$

where ϕ and ψ are arbitrary functions and R denotes $\{(c-z)^2+x^2+y^2\}^{\frac{1}{2}}$, satisfies the equation

$$\nabla^2 u+\kappa^2 v=0.$$

III. *The equation* $\nabla^2\phi=c^2\frac{\partial^2\phi}{\partial t^2}$.

12. This equation, the general equation for the propagation of sound in a homogeneous medium, can be integrated by making the equation a special instance of that which is considered in § 1. Writing

$$t=ic\tau,$$

we have

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}+\frac{\partial^2\phi}{\partial \tau^2}=0;$$

so that the solutions, obtained in §§ 1, 6, apply to this case by making $n=4$. The following theorem thus arises:

If p, q, r, s be any four arbitrary functions of a variable u subject to the single relation

$$p^2+q^2+r^2+s^2=0,$$

and if the variable u be determined as a function of four independent variables x, y, z, t by the equation

$$u=xp+yq+zr+\tau s,$$

where $t=ic\tau$; also, if v denote the quantity

$$\frac{\phi(u)}{1-xp'-yq'-zr'-\tau s'}+\psi(u),$$

where ϕ and ψ are arbitrary functions of u; then v satisfies the equation

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=c^2\frac{\partial^2\phi}{\partial t^2}.$$

13. Another solution of the equation can be obtained as follows. Taking the equation in the form

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}+\frac{\partial^2\phi}{\partial \tau^2}=0,$$

let four functions p, q, r, s of a variable u be determined (if possible) subject to the two conditions

$$p^2+q^2+r^2+s^2=0,$$
$$p'^2+q'^2+r'^2+s'^2=0;$$

conditions which involve the relations

$$\left.\begin{aligned} pp'+qq'+rr'+ss'&=0\\ pp''+qq''+rr''+ss''&=0\\ p'p''+q'q''+r'r''+s's''&=0\\ pp'''+qq'''+rr'''+ss'''&=0\end{aligned}\right\},$$

and consequently also the relation

$$\begin{vmatrix} p, & q, & r, & s\\ p', & q', & r', & s'\\ p'', & q'', & r'', & s''\\ p''', & q''', & r''', & s''' \end{vmatrix}=0,$$

deduced from the first, third, fourth, and sixth of these relations.

That quantities p, q, r, s actually exist, which satisfy the equations

$$p^2+q^2+r^2+s^2=0,$$
$$p'^2+q'^2+r'^2+s'^2=0,$$

can be seen as follows. The first equation is satisfied by writing

$$\frac{p}{s}=i\sin\theta\cos\phi,\quad \frac{q}{s}=i\cos\theta\cos\phi,\quad \frac{r}{s}=i\sin\phi,$$

where θ and ϕ are any arbitrary functions of u. These values give

$$p'^2+q'^2+r'^2=-s'^2-s^2(\theta'^2\cos^2\phi+\phi'^2),$$

and consequently

$$\theta'^2\cos^2\phi+\phi'^2=0,$$

whence
$$\sec\phi=\cos(\theta+\alpha),$$

where α is any constant. We therefore have

$$\frac{p}{s}=i\frac{\tan\theta}{\cos\alpha-\tan\theta\sin\alpha},\quad \frac{q}{s}=i\frac{1}{\cos\alpha-\tan\theta\sin\alpha},$$
$$\frac{r}{s}=\frac{\tan\theta\cos\alpha+\sin\alpha}{\cos\alpha-\tan\theta\sin\alpha},$$

where θ is any function of u. Hence writing

$$s = \phi(u) = \phi,$$
$$\tan\theta = \psi(u) = \psi,$$

the values of p, q, r, s satisfying the equations are

$$\left.\begin{aligned} p &= i\frac{\phi\psi}{\cos\alpha - \psi\sin\alpha} \\ q &= i\frac{\phi}{\cos\alpha - \psi\sin\alpha} \\ r &= \frac{\psi\cos\alpha + \sin\alpha}{\cos\alpha - \psi\sin\alpha}\phi \\ s &= \phi \end{aligned}\right\}.$$

By actually forming the derivatives, we find

$$p''^2 + q''^2 + r''^2 + s''^2 = 0.$$

This is a general consequence of the equations

$$p'^2 + q'^2 + r'^2 + s'^2 = 0,$$
$$p^2 + q^2 + r^2 + s^2 = 0;$$

and we therefore infer, by induction, that

$$\left(\frac{d^n p}{du^n}\right)^2 + \left(\frac{d^n q}{du^n}\right)^2 + \left(\frac{d^n r}{du^n}\right)^2 + \left(\frac{d^n s}{du^n}\right)^2 = 0.$$

Moreover, it is easy to verify that all the determinants

$$\left\|\begin{matrix} p'', & q'', & r'', & s'' \\ p', & q', & r', & s' \\ p, & q, & r, & s \end{matrix}\right\|$$

vanish.

14. Now let u be determined as a function of x, y, z, τ by the equation

$$u = xp + yq + zr + \tau s;$$

and let two other variables be defined by

$$\eta = xp' + yq' + zr' + \tau s',$$
$$\zeta = xp'' + yq'' + zr'' + \tau s''.$$

It will be convenient to use

$$\rho = xp''' + yq''' + zr''' + \tau s''';$$

but ρ is not a variable independent of u, η, ζ: for the determ-

inant of the coefficients of x, y, z, τ is zero, and therefore ρ is a function of the form

$$u_1 + u_2\eta + u_3\zeta,$$

where u_1, u_2, u_3 are functions of u, the precise forms of which are unimportant for the present purpose. We have

$$\frac{\partial u}{\partial x} = \frac{p}{1-\eta},$$

$$\frac{\partial \eta}{\partial x} = \frac{p}{1-\eta}\zeta + p',$$

$$\frac{\partial \zeta}{\partial x} = \frac{p}{1-\eta}\rho + p'';$$

and so for the derivatives with regard to y, z, τ.

We proceed to find what functions (if any) of the form $F(u, \eta, \zeta)$ satisfy the differential equation. When we substitute, and simplify by means of the equations among p, q, r, s and their derivatives, the equation is found to be identically satisfied.

The equation is thus satisfied whatever be the form of the function F; and we therefore have the theorem:

Any arbitrary function of u, η, ζ, where u, η, ζ are determined as functions of x, y, z, τ by the relations

$$\left.\begin{aligned} u &= xp + yq + zr + \tau s \\ \eta &= xp' + yq' + zr' + \tau s' \\ \zeta &= xp'' + yq'' + zr'' + \tau s'' \end{aligned}\right\},$$

and where p, q, r, s are arbitrary functions of u subject to the relations

$$\left.\begin{aligned} 0 &= p^2 + q^2 + r^2 + s^2 \\ 0 &= p'^2 + q'^2 + r'^2 + s'^2 \end{aligned}\right\},$$

is a solution of the equation

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 v}{\partial \tau^2} = 0.$$

Further Note on $\nabla^2 v = 0$.

15. Since the foregoing pages were in type, Dr. Hobson, to whom I mentioned some of these results, has reminded me of the known theorem that, if $f(x, y, z)$ denote any solution of Laplace's equation $\nabla^2 v = 0$, then

$$\frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}} f\left(\frac{x}{x^2+y^2+z^2}, \frac{y}{x^2+y^2+z^2}, \frac{z}{x^2+y^2+z^2}\right)$$

is also a solution of the equation. When this form of obtaining a new solution from a given one is taken into relation with the theorem of § 8, other new solutions are thus obtained.

The quantity u is determined as a function of x, y, z by the equation

$$au = xp(u) + yq(u) + zr(u);$$

and a solution of $\nabla^2 v = 0$ is given by

$$v = \phi(u) + \frac{\psi(u)}{1 - xp' - yq' - zr'}.$$

Let w denote the function which arises from u by changing x, y, z into $\frac{x}{R^2}, \frac{y}{R^2}, \frac{z}{R^2}$ respectively, R^2 denoting $x^2 + y^2 + z^2$; so that w is determined by the equation

$$aw = \frac{x}{R^2} p(w) + \frac{y}{R^2} q(w) + \frac{z}{R^2} r(w).$$

Then the application of the theorem quoted above leads to the result:—

A solution of Laplace's equation $\nabla^2 v = 0$ is furnished by taking

$$v = \frac{1}{R}\Phi(w) + \frac{1}{R}\,\frac{\Psi(w)}{1 - \frac{x}{R^2}p'(w) - \frac{y}{R^2}q'(w) - \frac{z}{R^2}r'(w)},$$

where Φ and Ψ are arbitrary functions, R^2 denotes $x^2 + y^2 + z^2$, the quantity w is determined by

$$aw = \frac{x}{R^2} p(w) + \frac{y}{R^2} q(w) + \frac{z}{R^2} r(w),$$

and p, q, r are arbitrary functions subject to the single relation

$$p^2 + q^2 + r^2 = 0.$$

The verification that the value of v given in this result actually satisfies Laplace's equation is rather long, but otherwise it is not difficult. It is my intention to discuss, in another paper, the relation of such solutions to one another.

ON AN IMPORTANT THEOREM WITH RESPECT TO THE OPERATION GROUPS OF ORDER p^{α}, p BEING ANY PRIME NUMBER.

By *G. A. Miller, Ph.D.*

FOR the sake of brevity, we shall represent an operation group of order p^{α} by P_{α}. It is well-known that P_{α} contains a subgroup of order p whose operators are commutative to every operator of P_{α}* that this subgroup is self-conjugate, and that each of the operators of any self-conjugate subgroup of order p is commutative to every operator of P_{α}.

It is also known that we may write all the operators of P_{α} in a column in such a way that the first p^{β} $(\beta=0, 1, 2, \ldots, \alpha-1)$ form a self-conjugate subgroup of P_{α}. In all the groups that we shall consider we shall suppose that the operators are arranged in this manner.

P_{α} has a p, 1 isomorphism to a group of order $p^{\alpha-1}$. For convenience we shall represent all the groups which we have placed in isomorphism to P_{α} or one of its subgroups by P'_{x}, p^{x} being the order of the group, while the subgroups of P_{α} will be represented by P_{y}, p^{y} being the order. Hence, we say that P_{x} is isomorphic to $P'_{\alpha-1}$.

Since each one of the first p operators of $P'_{\alpha-1}$ is commutative to all its operators, the second set of p operators of P_{α}, in this isomorphism, are transformed according to a group of order p or according to identity by all the operators of P_{α}. Hence we observe that P_{α} *contains a subgroup of order* $p^{\alpha-1}$, *such that each one of the operators of this subgroup is commutative to every one of its first* p^{2} *operators.*

We now use this subgroup $(P_{\alpha-1})$ in the same way as we used P_{α}, writing the p^{2} operators to which all its operators are commutative at the head, and observing that it has a p, 1 isomorphism to $P'_{\alpha-2}$. According to what has just been proved $P'_{\alpha-2}$ contains a $P'_{\alpha-3}$ such that all its operators are commutative to its first p^{2} operators. Since this $P'_{\alpha-3}$ corresponds to a subgroup of order $p^{\alpha-2}$ in the given $P_{\alpha-1}$, all the operators of this subgroup transform its p operators that follow its first p^{2} according to a group of order p or according to identity. Hence P_{α} *contains a subgroup of order* $p^{\alpha-2}$, *such that each one of the operators of this subgroup is commutative to every one of its first* p^{3} *operators.*

We now use this subgroup $(P_{\alpha-2})$ in the same way as we used $P_{\alpha-1}$, writing the p^{3} operators to which all its operators

* Sylow, *Mathematische Annalen*, Vol. v., p. 588.

are commutative at the head, and observing that it has a p, 1 isomorphism to $P'_{\alpha-4}$. This $P'_{\alpha-4}$ must be of the same type as the $P_{\alpha-1}$ in the preceding case; *i.e.* its first p^3 operators must be commutative to every one of its operators. Hence we may apply the result of the preceding case to it and observe that it contains a subgroup of order $p^{\alpha-6}$, such that all the operators of this subgroup are commutative to each one of its first p^3 operators.

To this $P'_{\alpha-6}$ there corresponds a subgroup of order $p^{\alpha-5}$ in the given $P_{\alpha-2}$. This $P_{\alpha-5}$ contains p^3 operators that are commutative to all its operators, and the first p operators that follow these are transformed according to a group of order p or according to identity by all its operators. Hence we observe that P_α *contains a subgroup of order* $p^{\alpha-6}$, *such that each one of the operators of this subgroup is commutative to every one of its first* p^4 *operators.*

We may now use this subgroup $(P_{\alpha-6})$ in the same way as we used $P_{\alpha-3}$, writing the p^4 operators to which all its operators are commutative at the head. The $P'_{\alpha-7}$ to which it has a p, 1 isomorphism is again of the same type as the preceding $P_{\alpha-4}$; *i.e.* its first p^3 operators are commutative to every one of its operators. Hence we may apply the preceding result to it and observe that it contains a subgroup of order $p^{\alpha-10}$, such that each one of the operators of this subgroup is commutative to every one of its first p^4 operators. In exactly the same way as before we observe that P_α *contains a subgroup of order* $p^{\alpha-10}$, *such that each one of the operators of this subgroup is commutative to every one of its first* p^5 *operators.*

Since this process of arguments can be continued indefinitely, we have the general

THEOREM. *A group of order* p^α *contains a subgroup of order* $p^{\alpha-\frac{\beta(\beta-1)}{2}}$, *such that each one of the operators of this subgroup is commutative to every one of the operators of a subgroup of order* p^β *that is contained in the given subgroup.*

Since $$\alpha-\frac{\beta(\beta-1)}{2} \gtreqless \beta,$$

we have $$\alpha \gtreqless \frac{\beta(\beta+1)}{2}.$$

Hence we obtain the

COROLLARY. *A group whose order exceeds* $p^{\frac{n(n-1)}{2}}$ *contains a commutative group of order* p^n.

When $n=3$ we observe that every group whose order exceeds p^3 contains a commutative group of order p^3. This special case of the given corollary is known. In a recent course of lectures Jordan suggested this corollary as a generalization of the given special case, but he did not give a complete proof.

Chicago, *October*, 1897.

INTERPOLATION TABLES.

By *P. J. Heawood.*

1. IN many cases where the first three terms of the Finite Difference formula $u_{n+x}=u_n+x\Delta u_n+\frac{x(x-1)}{2}\Delta^2 u_n+\dots$ would give with sufficient accuracy the intermediate values of a function to be tabulated when every tenth only had been directly calculated, it might be worth while to have assistance in supplying the intermediate values correct to the last place of decimals required, without the arithmetical work which the direct use of the formula involves. It is proposed to show that some very simple auxiliary tables will give material help in this direction, for cases where the final intervals of the correctly completed table do not vary too rapidly. Where (*e.g.*) to the last tabulated place there is, in each set of ten such intervals, a divergence of but a unit on either side of that which we may consider as the mean interval, we only require to be shown which intervals should be greater and which less than this mean, in order to supply them and complete the tabulation as required, the "mean interval" being the nearest integer to the average of the ten.

In the use of the above formula (u_1, u_2, u_3, ..., being the calculated numbers), the values intermediate to u_n, u_{n+1} would be got by giving to x in succession the values ·1, ·2, ·3, ..., ·9, and finally retaining the required number of decimal places only. Even the simplest case will show, however, that to get correctly even the numbers thus curtailed, we ought to know the calculated numbers to one place of decimals further than it is required finally to retain. Suppose that $u_n=114$, $u_{n+1}=136$, and that the nine intermediate values of u_{n+x} are required correct to the units place, second differences being zero. The calculated interval being 22 units, the intermediate intervals must average 2·2. 2 is therefore the "mean interval," and the ten intervals of the required set of numbers

will consist of eight 2's and two 3's, the larger intervals being the third and the eighth as is easily seen, that is *supposing that the above are the exact values of* u_n, u_{n+1}. If, however, 114 and 136 are only approximate values, representing, say, 114·4 and 136·4 (to take a case in which the first difference is precisely the same as before), the proper distribution of intervals will be quite different, the 3's being the first and the sixth. We can see in fact that the rejected tenths digit might be made an index to the right distribution of intervals, which even in this simple case cannot be correctly determined independently of it. And so for the general problem, which we proceed to consider, we shall suppose that* u_1, u_2, ... and their first and second differences have been formed to one place of decimals further than is finally required, the calculated numbers being taken in ascending order (which will make Δ positive, but will not affect the sign of Δ^2); and we shall refer to the last figures of the numbers thus expressed as the "last two" digits or the "final" digit, as the case may be, although we may speak of the last place finally required as the units place *of the interpolated numbers.*

2. *As thus calculated*, let $u_n = 10A + B$, $\Delta u_n = 100a + b$, $\Delta^2 u_n = c$; then, to the last place required, the nine values intermediate between u_n and u_{n+1} will be the integers nearest to the successive values of

$$A + \frac{B}{10} + m\left(a + \frac{b}{100}\right) - m\,(10 - m)\frac{c}{2000}:$$

for $m = 1, 2, \ldots, 9$. And so the m^{th} interval of the ten, being that between

$$\left[A + \frac{B}{10} + ma + y_m\right], \text{ and } \left[A + \frac{B}{10} + (m-1)\,a + y_{m-1}\right],$$

$\left(\text{where } y_m = \dfrac{2b - c}{200}\,m + \dfrac{c}{2000}\,m^2,\right.$ *and the square brackets* [ ] *are used to denote the nearest integers to the expressions they contain*$\left.\right)$, will differ from a, the mean interval, by just the difference of the integers $\left[\dfrac{B}{10} + y_m\right]$ and $\left[\dfrac{B}{10} + y_{m-1}\right]$, the precise amount of which will depend upon the digit B (unless indeed y_m, y_{m-1}

* Should a change of sign occur in the given series of numbers, the addition of a suitable constant will make them all positive.

contain the same number of tenths). If, as in the case with which we are chiefly concerned, *y_m, y_{m-1} differ by not more than unity, a set of values of B, equal in number to the difference of their tenths, will raise the integer corresponding to the greater a unit above that corresponding to the less, so that the interval in question would differ from a by unity for such a value of B; whereas if $B =$ one of the remaining digits, these two integers will be equal, and the interval will be a. Thus, if $y_m = 1{\cdot}32$, $y_{m-1} = {\cdot}75$ (the values for $m = 6$ when $b = 50$ and $c = 140$), the addition to both numbers of 2, 3, 4, 5, 6, or 7 tenths will bring up the integer nearest to the former to 2, while the nearest to the latter will be 1, so that we have a divergence of unity; whereas the addition of any other number of tenths would make these nearest integers both 2 or both 1.

Supposing at present that c is positive, the form of the expression for y_m as a quadratic function of m shows that it diminishes until $m = 5 - \dfrac{10b}{c}$, and then increases. This value of m may or may not lie between the limits $\frac{1}{2}$ and $9\frac{1}{2}$, but we have this at least as the most general case, in which $y_0, y_1, y_2, \ldots, y_{10}$ form a series of terms first diminishing and then increasing; and by what has been shown above (supposing still that no two consecutive y's differ by more than unity) the intervals corresponding to the former part of the series will be a or $a - 1$, to the latter a or $a + 1$, according to the value of B, the final digit of u_n, which we may call the "indicating digit." The sets of digits being determined and shown under the proper values of c and b, suitably distinguished, for which (1) intervals of the former class will be less, (2) intervals of the latter class will be greater than the mean, a glance at the proper table will show the arrangement of the intervals in any given case. In the subjoined example, taken from the table for $c = 140$, the lines are given which determine the "divergent" intervals for $b = 08$, 09, 10, the groups of digits showing the diminished intervals being marked off thus ________), and the rest showing the increased intervals: a set of more than five digits is shown by noting the *exceptions*, "*ex*567" being put instead of "4321098." Thus, taking the first line given, we should infer that if 7 (*e.g.*) were the indicating digit, the smaller intervals would be the first and third, the larger the seventh and ninth; if 2, the smaller would be the second, the larger the eighth and tenth.

* The y's may be negative, and the integers in question negative or zero; this will not affect the result, "greater" and "less" being understood algebraically.

Extract from Table for Second Difference = 140.

First Diff.	1	2	3	4	5	6	7	8	9	10	
08	67890	1234	567	8)	—	8	765	43210	98765	*ex*567	92
09	67890	1234	567	8)	—	87	654	3210	*ex*0128	*ex*456	91
10	67890	1234	56	7)	—	76	543	2109	*ex*9012	*ex*345	90

If two consecutive y's differ by more than unity, the only alteration in the result will be that the corresponding interval will then *always* differ from the mean by 1 at least, and may differ by 2 for certain values of the indicating digit. Such values of B may be shown as in the next example (taken from the same table as before) inside brackets after dots, thus ... ( ), the dots alone ... (which must be carefully distinguished from the blank—) indicating that the interval will differ by 1 from the mean for all values of the digit. Since we have $y_m - y_{m-1} = \frac{b}{100} + (2m-11)\frac{c}{2000}$, their difference will not exceed unity even for $b = 50$, until $c = 112$, and when $c = 150$ it will not exceed unity until $b = 33$. Even then they will not generally differ at once in their tenths, and the extra intervals will not come in so soon: not when $c = 150$, until $b = 43$, nor when $c = 140$ (as here) until $b = 47$.

48	6	— 6)	543	2109	87654	*ex*456	*ex*7	*ex*8	...(8)	52
49	6)	— 6	543	2109	*ex*9012	*ex*345	*ex*67	...	...(7)	51
50	6)	— 65	432	1098	*ex*8901	*ex*234	*ex*56	...	...(6)	50

3. Such is the general principle and arrangement of the proposed tables, but there are one or two important points in their formation to be noticed. First, it is easily seen that the digits in each division of a line will always, as in the above instances, proceed in regular succession from set to set, first upwards (from 6) in the former part, in the order 678901 and so on in rotation, and then in the latter part of the line downwards in the reverse order; (or downwards only, from 5, if the series of y's be entirely an increasing one, and all the sets of digits show increased intervals). For the last in a set of digits giving a *diminished* m^{th} interval, *taken in ascending order*, will be that which just does not raise the nearest integer to $\frac{B}{10} + y_m$ above the true value, and the next which just does raise it will be the first for which the succeeding interval is diminished (if at all). Similarly, it will be seen that the digits indicating *increased intervals* will rotate continually in *descending* order; also that the first digit of the latter series will be

identical with the last of the former, where both occur, and that the initial digit in the two cases will be as stated above.* Thus it is only necessary to know the *numbers* of digits in the several groups, which equal the numbers of tenths by which the successive y's differ, in order to write them down.

Moreover, when by an examination of the successive y's a table has been framed for a given value of c, from $b=00$ to $b=09$ inclusive, the rest of it can be written down at once; for if y'_m denotes what y_m becomes when $b+10$ takes the place of b, $y'_m = y_m + \frac{m}{10}$, so that each y interval will be less or more by just $\frac{1}{10}$ for the higher than for the lower value of b; and therefore in the remainder of the table the number of digits in each "diminishing group" will be one fewer, in each "increasing group" one more than in the corresponding place ten lines earlier. Thus, each "block" of ten b's after the first can be at once written down from the preceding, it being understood that in place of a *single* digit giving a *diminished* interval will come a blank, —, in the next block (indicating a mean interval whatever the digit), and in place of — a single digit for an *increased* interval in the next. All this is illustrated by the two portions of a table given above, which are for values of b differing by 40.

Again, it will be sufficient to tabulate for positive values of b only, from 00 to 50. For when the last two figures of First Difference exceed 50, so that b is a negative number (as implied in the formula), the intervals will be so related to those for the corresponding positive value of b, that they may be inferred from the same sets of digits. Suppose $\Delta v_n = \Delta u_n - 100r$, $v_n = u_n - 100\,(rn - s)$, where r, s are constant integers. If the Δv's are likewise positive, they will end in the *same* pairs of digits as the corresponding Δu's, while the corresponding smaller intervals of the two series of numbers (completed by interpolation) will just *differ* throughout by the constant r in the units place. But if r be large enough, and s too, so that the v's form a series of positive numbers in *descending* order of magnitude, their first differences, numerically, will end in pairs of digits *complementary* to those of the u's, while the intermediate intervals of the two completed series will also be complementary, in the sense that their *sums* will be r (which, r being an integer, will hold not only for the

* It must be borne in mind that if a calculated number ends in 5, the tables suppose the *lower* of the two integers between which it lies to be taken as its final value *if* Δ^2 *be* positive, the *higher if* Δ^2 *be* negative.

exact but also for the curtailed values). Thus, the intermediate intervals of the u's taken in ascending order, will be so related to those of the v's taken in descending order, that the greater intervals of either series will just coincide with the smaller of the other; and obviously the final digit of the greater of two u's will be identical with that of the smaller of the corresponding v's. This gives the required relation between the intervals for positive and negative values of b, and shows that if we place the complementary first difference figures from 50 to 100 at the ends of the lines as seen in the examples that have been given, the intervals for such a pair of figures will be shown in order by the sets of digits in line with it taken backwards, the sets marked______) giving now *greater* and the others the *smaller* intervals, and the indicating digit being now the final digit of the *higher* calculated number instead of the *lower*.*

Also the same tables will serve for negative Second Differences. For, let $w_n = 10t - u_n$, where t is a constant integer sufficiently large to make the w's positive: then $\Delta^2 w_n$ will be opposite in sign to $\Delta^2 u_n$, while the w's will form a *descending* series whose intervals (exact or approximate) are exactly the same numerically as those of the u's; and obviously the *greater* of two u's will have its final digit complementary to that of the smaller w, and *vice versa*. Thus, to obtain the intervals for a negative value of c from the table for the corresponding positive value, we must take as the indicating digit the complementary of the final of the higher or lower of the calculated numbers between which we are interpolating, according as the first difference ends in two digits *less* or *more* than 50; and the intervals will then be given just as before, but *in reverse order, from the higher to the lower* calculated number instead of from the lower to the higher.

Lastly, we may observe that owing to the smallness of the coefficient of c in the formula, as compared with that of b, we may use an approximate value of c. As will be seen below, no material error will be made by a divergence of 5 from its true value in the last calculated place; so that it will be sufficient to have tables for each value of c ending in zero, and to use the nearest in any given case. Thus, within the limits of roughly laid down at the first, we shall only require about 16 tables, taking them for values of c ranging from 00 to 150; for beyond this point intervals diverging by 2 from the mean will (we have seen) more and more frequently

* $b = 50$ may be treated in either way, according as we take for the mean interval the lower or higher of the two integers between which the average lies.

come in. There would be no difficulty indeed in carrying on the tables to any extent beyond this point, except for the more complicated notation that would be necessary for more varied intervals: but it might hardly be worth while to proceed much further, especially as higher values of Δ^2 would often occur in connection with values of Δ^3 too large to be neglected; (but with respect to the neglect of third differences, see below). However, the tables might be carried as far as $c = 330$, without the occurrence of any interval differing by *more* than 2 from the mean.

4. It only remains to calculate the exact limits of error in the values deduced by means of such a restricted series of tables applied to numbers not themselves exact but correct to the point supposed; and first consider that introduced into the estimate of a number by the possible divergence of 5 between the calculated value of c and that of the table we use. Since the largest value of its coefficient, that for $m = 5$, is ·0125, the extreme difference between the value of an intermediate number as calculated with any given value of c, and with that on which the nearest table is based, is only ·0625 or $\frac{1}{16}$, in the last place finally required, which is not much more than the limit of error of the calculated numbers themselves as carried correctly to the extra place ($=\frac{1}{20}$). We have still to determine the maximum error in the estimate of an intermediate value, due to these small possible divergences of $u_1, u_2, u_3, \ldots$ from the exact values. Since we have

$$u_n + x\Delta u_n + \frac{x(x-1)\Delta^2 u_n}{2} = u_n + x(u_{n+1} - u_n)$$
$$+ \frac{x(x-1)}{2}(u_{n+2} + u_n - 2u_{n+1}),$$

errors β, γ, δ in u_n, u_{n+1}, u_{n+2} will produce an error in the result of $\frac{(1-x)(2-x)}{2}\beta + x(2-x)\gamma - \frac{x(1-x)}{2}\delta$; and the coefficients under the signs shown being all positive, this will be greatest for a given value of x when $\beta = \gamma = -\delta = k$, the maximum value of each separate error. This gives to the whole the value $k(1 + x - x^2)$, which again has its maximum value when $x = \frac{1}{2}$, and then $= 1\frac{1}{4}k$, or $\frac{1}{16}$ (k being $\frac{1}{20}$). This is just the same as the maximum error from the other source, and the final result is that in the most unfavourable case possible, the divergence between the value that would be derived in any case from an exact application of the formula,

carried as far as second differences, and that corresponding to the estimates on which the tabular intervals of the nearest available table are based, cannot exceed $\frac{1}{8}$ in the last place of decimals required; and as the different errors may be expected to some extent to compensate each other, even when they individually reach anything like their extreme values, it is very rarely that the total would come nearly up to this utmost amount. Even so it would only be when the exact value lay within $\frac{1}{8}$ of midway between two integers, and the error was in the opposite direction, that the nearest integral value would be affected thereby; and finally the maximum error of the numbers obtained $=\frac{5}{8}$ (instead of $\frac{1}{2}$). If we had tables for every 5 of second difference, the error from the former source would be halved, and the maximum error in the reckoning would be only about $\frac{1}{11}$ in the last place retained instead of $\frac{1}{8}$.

It has been supposed so far that third differences may be safely neglected, and in general any small error due to this neglect may be considered as already allowed for under that due to the use of inexact values of second differences, but if these vary with any rapidity, it is worth noticing that a more accurate result will be obtained by taking for any interval the mean of the second differences which precede and follow it, rather than the one technically (but somewhat artificially and unsymmetrically) associated with it; this being indeed equivalent to taking some account of third differences as usually reckoned. For we have

$$\tfrac{1}{2}(\Delta^2 u_n + \Delta^2 u_{n-1}) = \Delta^2 u_n - \tfrac{1}{2}\Delta^3 u_{n-1} \text{ or } \Delta^2 u_n - \tfrac{1}{2}\Delta^3 u_n,$$

if we may suppose Δ^3 constant. Thus, the use of the mean value of Δ^2 is equivalent to the addition of a new term to the formula as hitherto used, namely, $\dfrac{x(1-x)}{2} \times \frac{1}{2}\Delta^3$, while the regular coefficient of Δ^3 is $\dfrac{x(1-x)}{2}\,\dfrac{2-x}{3}$, the latter factor of which ranges only from $\frac{1}{3}$ to $\frac{2}{3}$, with $\frac{1}{2}$ as its mean value. But the degree of precision of the proposed correction is much greater even than the ratio of these coefficients (ranging from $\frac{2}{3}$ to $\frac{4}{3}$) might suggest, since their values are extremely small when their ratio diverges most from unity. Thus, the "residual" coefficient of Δ^3 required to make up the term thus implicitly introduced to its true value, is $\dfrac{x(\frac{1}{2}-x)(1-x)}{6}$, the complete coefficient being $\dfrac{x(1-x)(2-x)}{6}$,

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and the maximum of the former is just one-eighth that of the latter, or $\frac{1}{72\sqrt{3}}$ (practically $=\frac{1}{125}$), in place of $\frac{1}{9\sqrt{3}}$ (which roughly $=\frac{1}{16}$); so that ~~if we use~~ mean second differences, the greatest error introduced by the neglect of third differences is only one-eighth of what it would otherwise be. Moreover, the symmetry introduced thereby sets at rest any questions that might arise owing to the divergent results given otherwise (when Δ^3 is not constant) for numbers taken in ascending or descending order. No alteration will be made in the extreme limit of error due to the small possible errors in u_1, u_2, ... by this variation on the usual formula, for it will be found that the resulting error in u_{n+x} due to the separate errors α, β, γ, δ in u_{n-1}, u_n, u_{n+1}, u_{n+2}, now

$$=\frac{(1-x)(4+x)}{4}\beta+\frac{x(5-x)}{4}\gamma-\frac{x(1-x)}{4}(\alpha+\delta),$$

and its greatest value equals the maximum of $(1+x-x^2)k=\frac{1}{16}$ as before. We have no occasion actually to form the mean second differences, as we have only to deal with round numbers in using the tables; only, on the above principle, we should select the number lying midmost between two values, rather than such as is nearest to one, as the heading of the table we use.

5. In conclusion two examples are appended of the practical working of the tables, in cases where second differences are positive and negative respectively, involving the use only of those small portions of that for $c=140$, which have been given as illustrations. First are shown the calculated numbers and their first and second differences (1); then (2) the numbers immediately given by them for reference to the tables, namely, Approximate Second Difference, Two Final Digits of First Difference, Indicating Digit ( ), and Mean Interval; then (3) the given numbers (curtailed) with the sets of ten intervals between them marked off by means of the tables as greater + or less − where they diverge from the mean, in accordance with which marking their values are supplied and the required tabulation completed. We have so far supposed the Δu's all of one sign. A change of their sign will, however, make no difference, if the rules based on the original supposition, that they are all positive, be borne in mind (*i.e.* as to indicating digit and order of intervals): the numbers may indeed be taken in either order, and the result will be quite unaffected if we use mean second differences.

(1)			(2)	(3)	(1)			(2)	(3)
11278			140	1128	11278			−140	1128
			08	12 −				91	16
	1308			1140		1591			1144
			(8)	13				(2)	17 +
12586		141		1153	12869		−142		1161
			13	13				16	16
	1449			1166		1449			1177
				12 −					16
14085		142		1178	14318		−141		1193
				13					16
	1591			1191		1308			1209
				14 +					16
15626				1205	15626				1225
				13					16
				1218					1241
				13					15 −
etc.				1231	etc.				1256
				14 +					16
				1245					1272
				14 +					15 −
			140	1259				−140	1287
			49	13 −				49	15 +
				1272					1302
			(6)	14				(2)	15 +
				1286					1317
			14	15 +				14	15 +
				1301					1332
				14					15 +
				1315					1347
				14					14
				1329					1361
				15 +					15 +
				1344					1376
				15 +					14
				1359					1390
				14					14
				1373					1404
				15 +					14
				1388					1418
				15 +					14
			140	1403				−140	1432
			91	16				08	14 +
				1419					1446
			(6)	15 −				(4)	13
				1434					1459
			16	16				13	14 +
				1450					1473
				15 −					13
				1465					1486
				16					13
				1481					1499
				16					13
				1497					1512
				16					13
				1513					1525
				17 +					13
				1530					1538
				16					12 −
				1546					1550
				17 +					13
				1563					1563

NOTE ON SURFACES WHOSE RADII OF CURVATURE ARE EQUAL AND OF THE SAME SIGN.

By *A. R. Forsyth.*

MONGE* gives a discussion of surfaces whose radii of curvature are equal and of the same sign. The differential equation of the second order, characteristic of such surfaces, is

$$\{(1+q^2)r - 2pqs + (1+p^2)t\}^2 = 4(rt-s^2)(1+p^2+q^2);$$

and it can be integrated by a process differing from the process adopted by Monge.

1. The equation may have an intermediary integral; if

$$u(x, y, z, p, q) = 0$$

denote such an integral, then the differential equation should be satisfied identically in virtue of

$$u_x + ru_p + su_q = 0, \quad u_y + su_p + tu_q = 0,$$

where u_x, u_y, u_p, u_q denote

$$\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}, \quad \frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}, \quad \frac{\partial u}{\partial p}, \quad \frac{\partial u}{\partial q}$$

respectively. Substituting then (say) for r and t their values in terms of s, viz.

$$r = -s\frac{u_q}{u_p} - \frac{u_x}{u_p}, \qquad t = -s\frac{u_p}{u_q} - \frac{u_y}{u_q},$$

the resulting form of the equation must be evanescent. The conditions, necessary and sufficient to secure this, are

$$(1+q^2)\frac{u_q}{u_p} + 2pq + (1+p^2)\frac{u_p}{u_q} = 0,$$

$$\frac{u_x}{u_q} + \frac{u_y}{u_p} = 0,$$

$$\left\{(1+q^2)\frac{u_x}{u_p} + (1+p^2)\frac{u_y}{u_q}\right\}^2 = 4\frac{u_x u_y}{u_p u_q}(1+p^2+q^2),$$

* *Application de l'analyse à la géométrie*, 5^{me} éd., pp. 196–211.

arising from the coefficients of s^2, s, 1 respectively. The third of these is satisfied in virtue of the first two; for assuming these two, we have

$$\left\{(1+q^2)\frac{u_x}{u_p}+(1+p^2)\frac{u_y}{u_q}\right\}^2-4\frac{u_xu_y}{u_pu_q}(1+p^2+q^2)$$

$$=\frac{u_x^2}{u_q^2}\left[\left\{(1+q^2)\frac{u_q}{u_p}-(1+p^2)\frac{u_p}{u_q}\right\}^2+4(1+p^2+q^2)\right]$$

$$=\frac{u_x^2}{u_q^2}\left[\left\{(1+q^2)\frac{u_q}{u_p}+(1+p^2)\frac{u_p}{u_q}\right\}^2-4p^2q^2\right]$$

$$=0.$$

Hence the equations that determine u are

$$\left.\begin{aligned}(1+q^2)u_q^2+2pq\,u_pu_q+(1+p^2)u_p^2=0\\ u_xu_p+u_yu_q=0\end{aligned}\right\}.$$

From the former, we have

$$(1+q^2)u_q+(pq\pm i\Delta)u_p=0,$$

where Δ denotes $(1+p^2+q^2)^{\frac{1}{2}}$; say this is

$$u_q+\mu u_p=0.$$

The second equation then is

$$u_x-\mu u_y=0,$$

that is,

$$\frac{\partial u}{\partial x}-\mu\frac{\partial u}{\partial y}+(p-\mu q)\frac{\partial u}{\partial z}=0,$$

which, taken with

$$\frac{\partial u}{\partial q}+\mu\frac{\partial u}{\partial p}=0,$$

constitutes a complete Jacobian system. For the Jacobi-Poisson condition is that the equation

$$-\frac{\partial\mu}{\partial q}\frac{\partial u}{\partial y}-\left(\mu+q\frac{\partial\mu}{\partial q}\right)\frac{\partial u}{\partial z}+\mu\left\{-\frac{\partial\mu}{\partial p}\frac{\partial u}{\partial y}+\left(1-q\frac{\partial\mu}{\partial p}\right)\frac{\partial u}{\partial z}\right\}=0,$$

or, what is the same thing,

$$\left(\frac{\partial\mu}{\partial q}+\mu\frac{\partial\mu}{\partial p}\right)\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)=0,$$

should either be satisfied in virtue of the two initial equations

or be an evanescent equation. Now, on actual substitution, we find

$$\frac{\partial \mu}{\partial q} + \mu \frac{\partial \mu}{\partial p} = 0;$$

and the required condition is therefore identically satisfied.

2. Conversely, any solution of the equations

$$\left.\begin{array}{l} u_x - \mu u_y = 0 \\ u_q + \mu u_p = 0 \end{array}\right\}$$

should lead to the original differential equation. Now from any solution, we have

$$u_x + r u_p + s u_q = 0, \quad u_y + s u_p + t u_q = 0;$$

hence

$$r u_p + s u_q - \mu (s u_p + t u_q) = 0.$$

and therefore

$$u_p (r - 2\mu s + \mu^2 t) = 0.$$

Assuming that the solution is not independent of p, we have

$$r - 2\mu s + \mu^2 t = 0.$$

To compare this with the original equation, write

$$\mu_1 = \frac{pq - i\Delta}{1 + q^2}, \quad \mu_2 = \frac{pq + i\Delta}{1 + q^2};$$

then

$$(r - 2\mu_1 s + \mu_1^2 t)(r - 2\mu_2 s + \mu_2^2 t)(1 + q^2)^2$$
$$= \{(1 + q^2) r - 2pqs + (1 + p^2) t\}^2 - 4 (rt - s^2)(1 + p^2 + q^2),$$

so that the original equation can be resolved into

$$r - 2\mu_1 s + \mu_1^2 t = 0,$$

$$r - 2\mu_2 s + \mu_2^2 t = 0.$$

Moreover, these equations are conjugate in the complex quantities: hence the solution of either can be deduced from the other, and therefore it is sufficient to consider one alone, say

$$r - 2\mu s + \mu^2 t = 0,$$

where μ now will be used to denote

$$\frac{pq - i\Delta}{1 + q^2}.$$

If the two conjugate equations be taken together, they give

$$\frac{r}{1+p^2}=\frac{s}{pq}=\frac{t}{1+q^2},$$

leading to a real locus of umbilici; but it is not necessary that the two equations be satisfied, for the original equation holds in virtue of either alone.*

3. The simplest of all the solutions is given by $r=0$, $s=0$, $t=0$, so that

$$z=ax+by+c,$$

a, b, c being arbitrary constants; the surface is a plane.

Turning to other solutions, we have the intermediary integral determined by

$$\left.\begin{aligned}\frac{\partial u}{\partial q}+\mu\frac{\partial u}{\partial p}&=0\\ \frac{\partial u}{\partial x}-\mu\frac{\partial u}{\partial y}+(p-\mu q)\frac{\partial u}{\partial z}&=0\end{aligned}\right\},$$

these two equations forming a complete system. As there are two equations in this complete system for u, and as there are five variables x, y, z, p, q, there are three functionally independent solutions; they can be obtained as follows.

It is manifest that, of the equation

$$\frac{\partial u}{\partial q}+\mu\frac{\partial u}{\partial p}=0,$$

three functionally independent solutions are x, y, z; and, because

$$\frac{\partial \mu}{\partial q}+\mu\frac{\partial \mu}{\partial p}=0,$$

it is manifest that μ is another solution,† which is functionally independent of the others. Consequently

$$u=f(x, y, z, \mu),$$

* As to this, see some remarks in Salmon's *Solid Geometry*, 3rd ed., p. 234, *note*.

† These solutions can be obtained also by considering the subsidiary equations

$$\frac{dx}{0}=\frac{dy}{0}=\frac{dz}{0}=\frac{dp}{\mu}=\frac{dq}{1}=\frac{du}{0}.$$

where f is any function, is the most general solution of the equation

$$\frac{\partial u}{\partial q} + \mu \frac{\partial u}{\partial p} = 0;$$

the form of f must be limited so as to satisfy

$$\frac{\partial u}{\partial x} - \mu \frac{\partial u}{\partial y} + (p - \mu q)\frac{\partial u}{\partial z} = 0,$$

that is, we must have

$$\frac{\partial f}{\partial x} - \mu \frac{\partial f}{\partial y} + (p - \mu q)\frac{\partial f}{\partial z} = 0.$$

Now $$\mu = \frac{pq - i\Delta}{1 + q^2};$$

from this it is easy to deduce that

$$\left.\begin{aligned} p &= \mu q + i(1+\mu^2)^{\frac{1}{2}} \\ \Delta &= i\mu + q(1+\mu^2)^{\frac{1}{2}} \end{aligned}\right\}.$$

so that the equation for f is

$$\frac{\partial f}{\partial x} - \mu \frac{\partial f}{\partial y} + i(1+\mu^2)^{\frac{1}{2}}\frac{\partial f}{\partial z} = 0.$$

The subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-\mu} = \frac{dz}{i(1+\mu^2)^{\frac{1}{2}}} = \frac{d\mu}{0} = \frac{df}{0};$$

three independent solutions of these are manifestly

$$\mu = u_1,$$
$$y + \mu x = u_2,$$
$$z - ix(1+\mu^2)^{\frac{1}{2}} = u_3,$$

and any function of u_1, u_2, u_3 is a solution of the equation.

4. It is known that, when three functionally independent solutions of the subsidiary system of an equation

$$Rr + Ss + Tt + U(rt - s^2) = V$$

have been obtained, then if the elimination of p and q between

$$u_1 = a,\ u_2 = b,\ u_3 = c$$

leads to an equation

$$F(x, y, z, a, b, c) = 0,$$

this equation gives an integral; and the integral is at once generalisable by an established theorem. The result does not, however, apply to the equation $Rr + Ss + Tt = 0$; for the elimination of p and q, being here the elimination of μ, leads to two equations instead of one, and neither of them is of the required form.

5. Accordingly, we proceed otherwise. It is clear that

$$y + \mu x = \phi(\mu),$$

where ϕ is arbitrary, is a solution of the subsidiary system, and it is easily verified to be an intermediary integral. For we have

$$\mu + \{x - \phi'(\mu)\}\left(\frac{\partial\mu}{\partial p} r + \frac{\partial\mu}{\partial q} s\right) = 0,$$

$$1 + \{x - \phi'(\mu)\}\left(\frac{\partial\mu}{\partial p} s + \frac{\partial\mu}{\partial q} t\right) = 0,$$

and therefore

$$\frac{\partial\mu}{\partial p} r + \frac{\partial\mu}{\partial q} s = \mu\left(\frac{\partial\mu}{\partial p} s + \frac{\partial\mu}{\partial q} t\right),$$

that is,
$$r - 2\mu s + \mu^2 t = 0.$$

Similarly, it is clear that

$$z - ix(1 + \mu^2)^{\frac{1}{2}} = \psi(\mu),$$

where ψ is arbitrary, is a solution of the subsidiary system; and for this also the verification, as to its being an intermediary integral, is simple. It thus appears that there are two distinct intermediary integrals, viz.

$$y + \mu x - \phi(\mu) = 0,$$

$$z - ix(1 + \mu^2)^{\frac{1}{2}} - \psi(\mu) = 0,$$

both of a general type: a question arises, can they coexist? The Poisson-Jacobi condition for coexistence is that the quantity

$$\begin{aligned}\{x - \phi'(\mu)\}\frac{\partial\mu}{\partial p}\{p - i(1 + \mu^2)^{\frac{1}{2}}\}& \\ + \{ix\mu(1 + \mu^2)^{-\frac{1}{2}} + \psi'(\mu)\}\frac{\partial\mu}{\partial p}\mu& \\ + \{x - \phi'(\mu)\}\frac{\partial\mu}{\partial q} q + \{ix\mu(1 + \mu^2)^{-\frac{1}{2}} + \psi'(\mu)\}\frac{\partial\mu}{\partial q}&\end{aligned}$$

should vanish. In this expression, the quantity which multiplies $x - \phi'(\mu)$ is

$$= q\frac{\partial\mu}{\partial q} + \{p - i(1+\mu^2)^{\frac{1}{2}}\}\frac{\partial\mu}{\partial p} = q\left(\frac{\partial\mu}{\partial q} + \mu\frac{\partial\mu}{\partial p}\right) = 0;$$

and the quantity which multiplies $ix\mu(1+\mu^2)^{-\frac{1}{2}} + \psi'(\mu)$ is

$$= \frac{\partial\mu}{\partial q} + \mu\frac{\partial\mu}{\partial p} = 0.$$

The condition is satisfied; and the two intermediary integrals therefore coexist.

6. When we take the equations together, the quantity μ can be regarded as eliminable: when elimination is effected, the result is an equation between x, y, z, involving two arbitrary functional forms. *This equation is the most general integral of the differential equation.* To establish this result, it is sufficient to shew that the differential equation is satisfied: the occurrence of the two arbitrary functions characterises the class of integral.

Of course, for purposes of elimination, μ is any eliminable quantity; and for purposes of obtaining the derivatives of z, μ is not necessarily the former combination of p and q. In fact, from the first of the equations

$$y + \mu x - \phi(\mu) = 0,$$

μ is equal to a function of x and y, such that

$$\mu + \{x - \phi'(\mu)\}\frac{\partial\mu}{\partial x} = 0, \quad 1 + \{x - \phi'(\mu)\}\frac{\partial\mu}{\partial y} = 0;$$

so that, from the second equation

$$z = ix(1+\mu^2)^{\frac{1}{2}} + \psi(\mu),$$

we have

$$p = i(1+\mu^2)^{\frac{1}{2}} + \{ix\mu(1+\mu^2)^{-\frac{1}{2}} + \psi'(\mu)\}\frac{\partial\mu}{\partial x}$$

$$= i(1+\mu^2)^{\frac{1}{2}} - \mu\frac{ix\mu(1+\mu^2)^{-\frac{1}{2}} + \psi'(\mu)}{x - \phi'(\mu)},$$

and similarly

$$q = -\frac{ix\mu(1+\mu^2)^{-\frac{1}{2}} + \psi'(\mu)}{x - \phi'(\mu)}.$$

Hence $\quad p = \mu q + i(1+\mu^2)^{\frac{1}{2}}$,

so that, when μ is expressed in terms of quantities other than x and y, we have

$$\mu = \frac{pq - i\Delta}{1+q^2}.$$

Again, from

$$p = \mu q + i(1+\mu^2)^{\frac{1}{2}},$$

we have

$$r - \mu s = \{q + i\mu(1+\mu^2)^{-\frac{1}{2}}\}\frac{\partial \mu}{\partial x}$$

$$= -\mu \frac{q + i\mu(1+\mu^2)^{-\frac{1}{2}}}{x - \phi'(\mu)},$$

and

$$s - \mu t = \{q + i\mu(1+\mu^2)^{-\frac{1}{2}}\}\frac{\partial \mu}{\partial y}$$

$$= -\frac{q + i\mu(1+\mu^2)^{-\frac{1}{2}}}{x - \phi'(\mu)};$$

hence $\quad r - \mu s = \mu(s - \mu t)$,

and therefore the differential equation is satisfied by the equation that results from the elimination of μ between the equations

$$\left.\begin{aligned} y + \mu x &= \phi(\mu) \\ z - ix(1+\mu^2)^{\frac{1}{2}} &= \psi(\mu) \end{aligned}\right\}.$$

7. One remark may be made in passing. The original equation can be satisfied either by

$$r - 2\mu_1 s + \mu_1^2 t = 0,$$

or by

$$r - 2\mu_2 s + \mu_2^2 t = 0.$$

Of the former, an intermediary integral is given by

$$\mu_1 = \text{constant};$$

and of the latter, an intermediary integral is given by

$$\mu_2 = \text{constant}.$$

It is simple to verify that these two intermediary integrals can co-exist. When taken together, they give

$$p = \text{constant} = a, \quad q = \text{constant} = b,$$

and therefore $$z = ax + by + c.$$

The equation of a plane accordingly is a solution.

Further, this equation involves three arbitrary constants; and it can therefore be generalised by using Imschenetsky's method. The resulting solution is found to be equivalent to the foregoing general solution, deducible from it by the (tangential) transformation originally due to Legendre.

8. Let θ denote $\mu + (1+\mu^2)^{\frac{1}{2}}$, so that

$$-\frac{1}{\theta} = \mu - (1+\mu^2)^{\frac{1}{2}}.$$

Then μ being a function of θ, we can replace the two equations determining our surface by combinations of the form

$$\left.\begin{aligned} y + iz + x\theta &= \phi(\mu) + i\psi(\mu) = F(\theta) \\ y - iz - x\frac{1}{\theta} &= \phi(\mu) - i\psi(\mu) = G(\theta) \end{aligned}\right\},$$

where F and G are arbitrary functions, independent of one another, because ϕ and ψ are independent arbitrary functions; this form may be regarded as giving *the integral of the equation.*

One simple case arises by taking

$$F(\theta) = A + B\theta, \quad G(\theta) = A' + B'\frac{1}{\theta},$$

where $$A = b + ic, \quad B = R + a,$$

$$A' = b - ic, \quad B' = R - a,$$

and a, b, c, R all are real; the elimination of θ leads to the equation

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2$$

as a surface that possesses the required property. According to Monge, it is the only case in which the solution is real.

ON PLANE EQUIPOTENTIAL SURFACES.

By *W. Burnside.*

1. This paper was suggested by Prof. Forsyth's application of his new method of integrating partial differential equations to the case of Laplace's equation, given on pp. 100–109 of the present volume.

The relation in which the second solution, which Prof. Forsyth gives (p. 107), stands to the first might easily be missed. This second solution is, in fact, the differential coefficient of any arbitrary function of the first with respect to either x, y, or z. Starting from this observation, I have endeavoured to determine the complete series of solutions which stand to Prof. Forsyth's first solution in the same relation, in which the system of harmonics of integral negative degree stand to r^{-1}.

I have also thrown Prof. Forsyth's first solution, so far as it relates to Laplace's equation, into a geometrical form, by taking as my starting point the determination of all systems of plane equipotential surfaces.

2. Let $$lx+my+nz+a=0 \quad \text{..................(i)}$$

be a system of plane equipotential surfaces; l, m, n being functions of the potential u, and a being a constant.

Then $$(l'x+m'y+n'z)\frac{\partial u}{\partial x}+l=0,$$

or $$\frac{\partial u}{\partial x}=-\frac{l}{D};$$

and

$$\frac{\partial^2 u}{\partial x^2}=-\frac{l'}{D}\frac{\partial u}{\partial x}+\frac{l}{D^2}\left(l'+\{l''x+m''y+n''z\}\frac{\partial u}{\partial x}\right)$$
$$=\frac{2ll'}{D^2}-\frac{l^2(l''x+m''y+n''z)}{D^3}.$$

Hence

$$0=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$$
$$=\frac{2(ll'+mm'+nn')}{D^2}-\frac{(l^2+m^2+n^2)(l''x+m''y+n''z)}{D^3}.$$

It follows that, if

$$l^2 + m^2 + n^2 \neq 0,$$

then
$$\frac{2(ll' + mm' + nn')}{l^2 + m^2 + n^2} = \frac{l''x + m''y + n''z}{l'x + m'y + n'z}$$

must be identically satisfied by all systems of values of u, x, y, z which satisfy (i).

Hence, eliminating x,

$$\frac{2(ll' + mm' + nn')}{l^2 + m^2 + n^2} = \frac{al'' + y(l''m - lm'') + z(l''n - ln'')}{al' + y(l'm - lm') + z(l'n - ln')}$$

must be satisfied by all values of u, y, and z.

If a is not zero, this involves

$$\frac{2(ll' + mm' + nn')}{l^2 + m^2 + n^2} = \frac{l''}{l'} = \frac{l''m - lm''}{l'm - lm'} = \frac{l''n - ln''}{l'n - ln'};$$

and eliminating y or z, each of these fractions is also equal to

$$\frac{m''}{m'} = \frac{n''}{n'} = \frac{m''n - mn''}{m'n - mn'}.$$

From these equations a simple integration gives

$$l = A + B\tan ku, \quad m = A' + B'\tan ku, \quad n = A'' + B''\tan ku,$$

where k and the A's and B's are arbitrary constants connected by the relations

$$A^2 + A'^2 + A''^2 = B^2 + B'^2 + B''^2 \text{ and } AB + A'B' + A''B'' = 0;$$

with

$$l = Cu^{-1}, \quad m = C'u^{-1}, \quad n = C''u^{-1},$$

as a particular case.

When a is zero, the relations are

$$\frac{2(ll' + mm' + nn')}{l^2 + m^2 + n^2} = \frac{l''m - lm''}{l'm - lm'} = \frac{m''n - mn''}{m'n - mn'} = \frac{n''l - nl''}{n'l - nl'},$$

and the integral agrees with the first of the two preceding cases.

Hence, unless $l^2 + m^2 + n^2 = 0$, the only systems of plane equipotential surfaces are, as is otherwise well known, a system of planes passing through a straight line, with a system of parallel planes as a particular case.

On the other hand if l, m, n are functions of u, such that

$$l^2 + m^2 + n^2 = 0,$$

no further condition is necessary to ensure that the system of planes (i) shall be a system of equipotential surfaces. Any singly-infinite system of planes therefore, each of which touches the absolute, *i.e.* the imaginary conic at infinity.

$$x^2 + y^2 + z^2 = 0,$$

is a system of equipotential surfaces.

3. The potential u is given by the simultaneous equations

$$lx + my + nz + a = 0 \text{ ....(ii)},$$

$$l^2 + m^2 + n^2 = 0 \text{ ....(iii)},$$

where l, m, n are to be regarded as otherwise undetermined functions of u. If v is any arbitrarily chosen function of u, the form of these equations is unaltered when we regard l, m, n as functions of v instead of u, and v is therefore a potential.

Conversely, if we start by assuming that u is a potential, such that any arbitrary function of u is also a potential, we are led back at once to the above equations. For

$$\frac{\partial}{\partial x} f(u) = f'(u) \frac{\partial u}{\partial x},$$

$$\frac{\partial^2}{\partial x^2} f(u) = f'(u) \frac{\partial^2 u}{\partial x^2} + f''(u) \left(\frac{\partial u}{\partial x}\right)^2,$$

Hence, if $\quad \nabla^2 u = 0$ and $\nabla^2 f(u) = 0,$

then $\quad f''(u) \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right] = 0;$

and hence, $f(u)$ being an arbitrary function,

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 0.$$

The general solution of this equation is

$$u = F(lx + my + nz),$$

where $$0 = l^2 + m^2 + n^2,$$

l, m, n being functions of u; and if the condition

$$\nabla^2 u = 0$$

is now introduced it leads to

$$F'' = 0,$$

so that F must be a linear function.

4. Returning now to the equations (ii) and (iii), we may write without loss of generality

$$\frac{l}{n} = \tfrac{1}{2}\left(U - \frac{1}{U}\right),$$

and then $$\frac{m}{n} = \frac{i}{2}\left(U + \frac{1}{U}\right),$$

U being an arbitrarily chosen function of u.

The equation (ii) then becomes

$$(U^2 - 1)\,x + i\,(U^2 + 1)\,y + 2\,Uz + 2a\,Un = 0.$$

If v is the particular function of u which we wish to define, we may take $U = v$, and $2a\,Un$ then becomes a function of v. The potential v is therefore given explicitly by the equation

$$f(v) = (v^2 - 1)\,x + i\,(v^2 + 1)\,y + 2vz \;\ldots\ldots\ldots\text{(iv)}.$$

From v, regarded as a fundamental solution of Laplace's equation, we may construct an infinite series of derived solutions of the form

$$\frac{\partial^{l+m+n}}{\partial x^l \partial y^m \partial z^n}\, G_{l,m,n}(v),$$

where $G_{l,m,n}(v)$ is an arbitrary function of v; exactly as, in the theory of spherical harmonics of integral degree, the successive harmonics are formed by differentiating r^{-1}; and here, as in the theory referred to, the question arises of how far these solutions are linearly independent (the functions of v involved being, of course, considered quite arbitrary).

If we use the abbreviations

$$p = v^2 - 1,\quad q = i\,(v^2 + 1),\quad r = 2v,$$

$$D = f'(v) - 2vx - 2ivy - 2z,$$

the equation (iv) gives on differentiation,

$$\frac{\partial v}{\partial x}=\frac{p}{D},\quad \frac{\partial v}{\partial y}=\frac{q}{D},\quad \frac{\partial v}{\partial z}=\frac{r}{D}.$$

If then G is an arbitrary function of v, and if we denote differential coefficients with respect to v by accents, we have

$$\frac{\partial G}{\partial x}=\frac{pG'}{D},\quad \frac{\partial G}{\partial y}=\frac{qG'}{D},\quad \frac{\partial G}{\partial z}=\frac{rG'}{D}.$$

Hence G_1, G_2, G_3, being three arbitrary functions of v, we cannot regard

$$\frac{\partial G_1}{\partial x},\quad \frac{\partial G_2}{\partial y},\quad \frac{\partial G_3}{\partial z}$$

as linearly independent; for we have in fact

$$\frac{\partial G_1}{\partial x}=\frac{\partial G_2}{\partial y},$$

if
$$pG_1'=qG_2';$$

i.e. whatever function G_1 may be, we can always choose G_2 so that $\frac{\partial G_1}{\partial x}$ may be expressed in the form $\frac{\partial G_2}{\partial y}$.

(Prof. Forsyth's second independent solution can evidently be written in any one of the three equivalent forms

$$\frac{\partial G_1}{\partial x},\quad \frac{\partial G_2}{\partial y},\quad \text{or}\ \frac{\partial G_3}{\partial z}.)$$

The solutions G and $\frac{\partial G_1}{\partial x}$ are, however, clearly independent, for $\frac{\partial G_1}{\partial x}$ is infinite at every point of the developable surface obtained by eliminating v between

$$f(v)-(v^2-1)\,x-2i\,(v^2+1)\,y-2vz=0,$$

and
$$f'(v)-2vx-2ivy-2z=0.$$

For a similar reason $\frac{\partial^{l+m+n}}{\partial x^l\partial y^m\partial z^n}G$ and $\frac{\partial^{l'+m'+n'}}{\partial x^{l'}\partial y^{m'}\partial z^{n'}}G'$ are independent if $l+m+n$ is not equal to $l'+m'+n'$.

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5. It remains to consider how many of the solutions $\dfrac{\partial^{l+m+n}}{\partial x^l \partial y^m \partial z^n} G$ are independent when $l+m+n$ has a given value; and the following considerations show that there is then only one independent solution.

Suppose that the relation

$$\frac{\partial^{l+m+n}}{\partial x^l \partial y^m \partial z^n} G = \left(\frac{1}{D}\frac{\partial}{\partial v}\right)^{l+m+n-1} \frac{p^l q^m r^n G'}{D}$$

has been proved for all values of $l+m+n$ less than some given integer; we shall shew that it is true when the value of $l+m+n$ is increased by 1, and therefore that it is true generally. The symbol $\dfrac{\partial}{\partial x}$ on the left-hand side denotes a partial differentiation with respect to x, where v is regarded as depending on x, and the symbol $\dfrac{\partial}{\partial v}$ on the right-hand side denotes a partial differentiation with respect to v on the supposition that v, x, y and z are independent. We shall require a symbol to denote partial differentiation of a function of v, x, y, z with respect to x on the supposition that the four variables are independent, and we shall use $\dfrac{\delta}{\delta x}$ in this sense. With this notation, if H is any function of v, x, y, z, then

$$\begin{aligned}
\frac{\partial}{\partial x}\frac{1}{D}\frac{\partial H}{\partial v} &= \frac{\partial}{\partial v}\left(\frac{1}{D}\frac{\partial H}{\partial v}\right)\frac{\partial v}{\partial x} + \frac{\delta}{\delta x}\left(\frac{1}{D}\frac{\partial H}{\partial u}\right) \\
&= \frac{p}{D}\frac{\partial}{\partial v}\left(\frac{1}{D}\frac{\partial H}{\partial v}\right) + \frac{\delta}{\delta x}\left(\frac{1}{D}\frac{\partial H}{\partial v}\right) \\
&= \frac{1}{D}\frac{\partial}{\partial v}\left(\frac{p}{D}\frac{\partial H}{\partial v}\right) - \frac{p'}{D^2}\frac{\partial H}{\partial v} \\
&\quad + \frac{p'}{D^2}\frac{\partial H}{\partial v} + \frac{1}{D}\frac{\delta}{\delta x}\frac{\partial H}{\partial v} \\
&= \frac{1}{D}\frac{\partial}{\partial v}\left(\frac{p}{D}\frac{\partial H}{\partial v} + \frac{\delta H}{\delta x}\right) \\
&= \left(\frac{1}{D}\frac{\partial}{\partial v}\right)\frac{\partial H}{\partial x}.
\end{aligned}$$

Hence, from the above assumption,

$$\frac{\partial^{l+m+n+1}}{\partial x^{l+1}\partial y^m\partial z^n}G=\left(\frac{1}{D}\frac{\partial}{\partial v}\right)^{l+m+n-1}\left(\frac{p}{D}\frac{\partial}{\partial v}+\frac{\delta}{\delta x}\right)\frac{p^l q^m r^n G'}{D}.$$

Also $$\frac{p}{D}\frac{\partial}{\partial v}\frac{p^l q^m r^n G'}{D}=\frac{1}{D}\frac{\partial}{\partial v}\frac{p^{l+1}q^m r^n G'}{D}-\frac{p'p^l q^m r^n G'}{D^2}$$

$$=\frac{1}{D}\frac{\partial}{\partial v}\frac{p^{l+1}q^m r^n G'}{D}-\frac{\delta}{\delta x}\frac{p^l q^m r^n G'}{D};$$

and therefore, finally

$$\frac{\partial^{l+m+n+1}}{\partial x^{l+1}\partial y^m\partial z^n}G=\left(\frac{1}{D}\frac{\partial}{\partial v}\right)^{l+m+n}\frac{p^{l+1}q^m r^n G'}{D}.$$

The relation is thus proved, and it follows at once that, when $l+m+n$ is given, every solution of the form

$$\frac{\partial^{l+m+n}}{\partial x^l\partial y^m\partial z^n}G,$$

where G is any function of v, can also be expressed in the form

$$\left(\frac{1}{D}\frac{\partial}{\partial v}\right)^{l+m+n}H.$$

Hence the most general solution that can be constructed by differentiation from the fundamental solution v is

$$\sum_0^\infty\left(\frac{1}{D}\frac{\partial}{\partial v}\right)^n H_n,$$

H_n being, for each value of n, an arbitrary function of v.

If we write $\phi+i\psi$ for v, where ϕ and ψ are real, the real part of the most general solution just obtained may be written in the form

$$\sum_0^\infty\left(\frac{\partial}{\partial x}\right)^n\Phi_n,$$

where for each value of n, Φ_n is an arbitrary (real) conjugate function of the two real functions ϕ and ψ.

6. Returning again to the fundamental solution itself given by

$$f(v)=(v^2-1)x+i(v^2+1)y+2vz,$$

and writing in it

$$v = \phi + i\psi,$$

$$f(v) = \Phi + i\Psi,$$

so that Φ and Ψ are conjugate functions of ϕ and ψ, the two conjugate potentials ϕ and ψ, as Prof. Forsyth calls them, are given by

$$\Phi = (\phi^2 - \psi^2 - 1)\,x - 2\phi\psi y + 2\phi z,$$

$$\Psi = 2\phi\psi x + (\phi^2 - \psi^2 + 1)\,y + 2\phi z.$$

When ϕ and ψ are constant, these are the equations of one or more straight lines. The equipotential surfaces corresponding to the two real conjugate potentials ϕ and ψ derived from any fundamental solution v, are therefore ruled surfaces, which intersect each other, necessarily at right angles, along their generators.

7. If we define a potential in non-Euclidean space as a homogeneous solution of degree zero of the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} + \epsilon \frac{\partial^2 V}{\partial t^2} = 0,$$

where for elliptic space ϵ is $+1$, and for hyperbolic ϵ is -1,* the problem of determining all systems of plane equipotential surfaces is closely analogous to that for ordinary space.

For elliptic space we get either a system of real planes all passing through a straight line, or a system of imaginary planes all touching the absolute, *i.e.* the imaginary quadric

$$x^2 + y^2 + z^2 + t^2 = 0.$$

Such a system is given by

$$lx + my + nz + t = 0,$$

* This is a suggestion of Prof. Klein. In ordinary space $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the only differential operator of the second order which is invariant for the group of motions; and the same is true for $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \epsilon \frac{\partial^2}{\partial t^2}$ in non-Euclidean space.

where $$l^2+m^2+n^2+1=0;$$

and if l, m, n are functions of u satisfying these conditions, then u is a potential.

In hyperbolic space we have either a system of planes passing through a straight line, or a system of planes all perpendicular to one straight line, or finally a system of (either ideal or imaginary) planes all touching the absolute, viz. the quadric

$$x^2+y^2+z^2-t^2=0.$$

Such a system is given by

$$lx+my+nz+t=0,$$

where $$l^2+m^2+n^2-1=0;$$

and again, if l, m, n are functions of u satisfying these equations, then u is a potential.

It is to be noticed that in this case l, m, n may be real functions of u, and the system of planes regarded as planes of ordinary projective space will then be a real system of planes. But for values of $x:y:z:t$ which correspond to real points in non-Euclidean space, *i.e.* for points of ordinary projective space which lie inside the quadric

$$x^2+y^2+z^2-t^2=0,$$

the value of u defined by the above equations will still be imaginary.

This points to a notable distinction between the potentials of ordinary and of hyperbolic space. While in ordinary space a potential which is finite at all finite points is infinite at all points at infinity, in hyperbolic space a potential may very well be finite at all finite points, and infinite at only one point at infinity. As an example,

$$\frac{x^2+y^2+z^2-t^2}{(lx+my+nz-t)^2},$$

where l, m, n are constants, is a potential; and, if

$$l^2+m^2+n^2-1=0,$$

the only point within or upon the absolute at which it becomes infinite is the point $x:y:z:t=l:m:n:1$.

NOTE ON THE MOTION OF AN INCOMPRESSIBLE VISCOUS FLUID.

By *J. Brill, M.A.*

In the case of the motion of an incompressible viscous fluid under a conservative system of forces, it will be interesting to consider the question of the most general hypothesis that permits of the vortex lines moving with the fluid.

The differential equations that control the vortex motion are

$$\left.\begin{aligned}
\frac{\partial\xi}{\partial t}+u\frac{\partial\xi}{\partial x}+v\frac{\partial\xi}{\partial y}+w\frac{\partial\xi}{\partial z}&=\xi\frac{\partial u}{\partial x}+\eta\frac{\partial u}{\partial y}+\zeta\frac{\partial u}{\partial z}+\nu\nabla^2\xi\\
\frac{\partial\eta}{\partial t}+u\frac{\partial\eta}{\partial x}+v\frac{\partial\eta}{\partial y}+w\frac{\partial\eta}{\partial z}&=\xi\frac{\partial v}{\partial x}+\eta\frac{\partial v}{\partial y}+\zeta\frac{\partial v}{\partial z}+\nu\nabla^2\eta\\
\frac{\partial\zeta}{\partial t}+u\frac{\partial\zeta}{\partial x}+v\frac{\partial\zeta}{\partial y}+w\frac{\partial\zeta}{\partial z}&=\xi\frac{\partial w}{\partial x}+\eta\frac{\partial w}{\partial y}+\zeta\frac{\partial w}{\partial z}+\nu\nabla^2\zeta
\end{aligned}\right\}\ldots(1).$$

Now, introducing an unknown coefficient λ, we have

$$\begin{aligned}
\frac{\partial(\lambda\xi)}{\partial t}+u\frac{\partial(\lambda\xi)}{\partial x}+v\frac{\partial(\lambda\xi)}{\partial y}+w\frac{\partial(\lambda\xi)}{\partial z}
&=\lambda\left\{\frac{\partial\xi}{\partial t}+u\frac{\partial\xi}{\partial x}+v\frac{\partial\xi}{\partial y}+w\frac{\partial\xi}{\partial z}\right\}\\
&\quad+\xi\left\{\frac{\partial\lambda}{\partial t}+u\frac{\partial\lambda}{\partial x}+v\frac{\partial\lambda}{\partial y}+w\frac{\partial\lambda}{\partial z}\right\}.
\end{aligned}$$

Thus, if we assume

$$\frac{\nabla^2\xi}{\xi}=\frac{\nabla^2\eta}{\eta}=\frac{\nabla^2\zeta}{\zeta}=U,$$

and
$$\frac{\partial\lambda}{\partial t}+u\frac{\partial\lambda}{\partial x}+v\frac{\partial\lambda}{\partial y}+w\frac{\partial\lambda}{\partial z}+\nu\lambda U=0,$$

we have

$$\left.\begin{aligned}
\frac{\partial(\lambda\xi)}{\partial t}+u\frac{\partial(\lambda\xi)}{\partial x}+v\frac{\partial(\lambda\xi)}{\partial y}+w\frac{\partial(\lambda\xi)}{\partial z}&=\lambda\xi\frac{\partial u}{\partial x}+\lambda\eta\frac{\partial u}{\partial y}+\lambda\zeta\frac{\partial u}{\partial z}\\
\frac{\partial(\lambda\eta)}{\partial t}+u\frac{\partial(\lambda\eta)}{\partial x}+v\frac{\partial(\lambda\eta)}{\partial y}+w\frac{\partial(\lambda\eta)}{\partial z}&=\lambda\xi\frac{\partial v}{\partial x}+\lambda\eta\frac{\partial v}{\partial y}+\lambda\zeta\frac{\partial v}{\partial z}\\
\frac{\partial(\lambda\zeta)}{\partial t}+u\frac{\partial(\lambda\zeta)}{\partial x}+v\frac{\partial(\lambda\zeta)}{\partial y}+w\frac{\partial(\lambda\zeta)}{\partial z}&=\lambda\xi\frac{\partial w}{\partial x}+\lambda\eta\frac{\partial w}{\partial y}+\lambda\zeta\frac{\partial w}{\partial z}
\end{aligned}\right\}$$

................(2).

These last three equations express that the curves defined by the differential equations

$$\frac{dx}{\lambda\xi}=\frac{dy}{\lambda\eta}=\frac{dz}{\lambda\zeta},$$

i.e. the vortex lines, move with the fluid. Further, it is evident that the assumptions that we have introduced are the most general that permit of this type of motion.

Taking some particular phase of the motion for the initial state, we will draw through the vortex lines two families of surfaces, whose parameters are m and β, so chosen that

$$2\xi_0=\frac{\partial(m,\beta)}{\partial(y_0,z_0)},\quad 2\eta_0=\frac{\partial(m,\beta)}{\partial(z_0,x_0)},\quad 2\zeta_0=\frac{\partial(m,\beta)}{\partial(x_0,y_0)}.$$

We know that this is possible, since we have

$$\frac{\partial\xi_0}{\partial x_0}+\frac{\partial\eta_0}{\partial y_0}+\frac{\partial\zeta_0}{\partial z_0}=0.$$

If the motion of the fluid be known, *i.e.* if u, v, w be known as functions of x, y, z, t, then the initial coordinates x_0, y_0, z_0 of the particle at (x, y, z) at the time t, can be obtained by solving the equations

$$\frac{\partial x_0}{\partial t}+u\frac{\partial x_0}{\partial x}+v\frac{\partial x_0}{\partial y}+w\frac{\partial x_0}{\partial z}=0,$$

$$\frac{\partial y_0}{\partial t}+u\frac{\partial y_0}{\partial x}+v\frac{\partial y_0}{\partial y}+w\frac{\partial y_0}{\partial z}=0,$$

$$\frac{\partial z_0}{\partial t}+u\frac{\partial z_0}{\partial x}+v\frac{\partial z_0}{\partial y}+w\frac{\partial z_0}{\partial z}=0.$$

The integrals obtained must, of course, satisfy the conditions $x_0=x$, $y_0=y$, $z_0=z$ for $t=0$.

Let these values be so determined, and substituted for x_0, y_0, z_0 in the expressions for m and β. These expressions will then determine the surfaces which are the loci, at the time t, of the particles of fluid which composed the original surfaces. They therefore determine two families of surfaces moving with the fluid, and since the parameter of each surface retains its original numerical value, we have

$$\frac{dm}{dt}=0,\quad \frac{d\beta}{dt}=0,$$

that is

$$\frac{\partial m}{\partial t} + u\frac{\partial m}{\partial x} + v\frac{\partial m}{\partial y} + w\frac{\partial m}{\partial z} = 0,$$

$$\frac{\partial \beta}{\partial t} + u\frac{\partial \beta}{\partial x} + v\frac{\partial \beta}{\partial y} + w\frac{\partial \beta}{\partial z} = 0.$$

Now consider a given pair of surfaces m and β. These surfaces move with the fluid, and therefore carry their vortex lines with them. The history of their line of intersection will be the history of a particular vortex line. Being the intersection of the two surfaces, it will always consist of the same particles; and, having been originally a vortex line, it will continue to be such.

Further, consider the vortex filament enclosed by the surfaces m, $m+dm$, β, $\beta+d\beta$. Let $d\sigma$ be the area of a cross section, and ω the average spin within that section, $d\sigma_0$ and ω_0 being the initial values of these quantities. Then if ds_0 be the original distance apart of two neighbouring particles whose distance is now ds, we have, on interpreting equations (2) after the manner of Helmholtz,

$$ds : ds_0 :: \lambda\omega : \lambda_0\omega_0.$$

Also, since the fluid is incompressible, we have

$$ds\, d\sigma = ds_0 d\sigma_0,$$

and therefore

$$\lambda\omega\,.\,d\sigma = \lambda_0\omega_0\,.\,d\sigma_0.$$

Also

$$\begin{aligned}2\omega_0 d\sigma_0 &= 2\,\{\xi_0 dy_0 dz_0 + \eta_0 dz_0 dx_0 + \zeta_0 dx_0 dy_0\}\\ &= \frac{\partial(m,\beta)}{\partial(y_0, z_0)}dy_0 dz_0 + \frac{\partial(m,\beta)}{\partial(z_0, x_0)}dz_0 dx_0 + \frac{\partial(m,\beta)}{\partial(x_0, y_0)}dx_0 dy_0,\\ &= dm\,d\beta,\end{aligned}$$

and therefore

$$2\frac{\lambda}{\lambda_0}\,\omega\, d\sigma = dm\, d\beta,$$

it being further to be remarked that dm and $d\beta$ retain their values throughout the motion. Thus we have

$$\begin{aligned}&2\frac{\lambda}{\lambda_0}\{\xi\, dy\, dz + \eta\, dz\, dx + \zeta\, dx\, dy\}\\ &\qquad = \frac{\partial(m,\beta)}{\partial(y, z)}dy\,dz + \frac{\partial(m,\beta)}{\partial(z, x)}dz\,dx + \frac{\partial(m,\beta)}{\partial(x, y)}dx\,dy;\end{aligned}$$

from which we immediately infer

$$2\xi = \frac{\lambda_0}{\lambda}\frac{\partial(m, \beta)}{\partial(y, z)}, \quad 2\eta = \frac{\lambda_0}{\lambda}\frac{\partial(m, \beta)}{\partial(z, x)}, \quad 2\zeta = \frac{\lambda_0}{\lambda}\frac{\partial(m, \beta)}{\partial(x, y)}.$$

By means of the relation

$$\frac{\partial\xi}{\partial x} + \frac{\partial\eta}{\partial y} + \frac{\partial\zeta}{\partial z} = 0,$$

we immediately deduce

$$\frac{\partial(\lambda_0/\lambda, m, \beta)}{\partial(x, y, z)} = 0,$$

from which it follows that

$$\frac{\lambda_0}{\lambda} = f(m, \beta, t).$$

Now we have

$$\frac{\partial\lambda}{\partial t} + u\frac{\partial\lambda}{\partial x} + v\frac{\partial\lambda}{\partial y} + w\frac{\partial\lambda}{\partial z} + \nu\lambda U = 0,$$

$$\frac{\partial\lambda_0}{\partial t} + u\frac{\partial\lambda_0}{\partial x} + v\frac{\partial\lambda_0}{\partial y} + w\frac{\partial\lambda_0}{\partial z} = 0,$$

and therefore

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)\log\frac{\lambda_0}{\lambda} = \nu U.$$

Thus writing

$$n = \log\frac{\lambda_0}{\lambda} = F(m, \beta, t),$$

we have

$$\frac{\partial n}{\partial t} = \frac{\partial F}{\partial t} + \frac{\partial n}{\partial m}\frac{\partial m}{\partial t} + \frac{\partial n}{\partial \beta}\frac{\partial \beta}{\partial t},$$

$$\frac{\partial n}{\partial x} = \frac{\partial n}{\partial m}\frac{\partial m}{\partial x} + \frac{\partial n}{\partial \beta}\frac{\partial \beta}{\partial x},$$

$$\frac{\partial n}{\partial y} = \frac{\partial n}{\partial m}\frac{\partial m}{\partial y} + \frac{\partial n}{\partial \beta}\frac{\partial \beta}{\partial y},$$

$$\frac{\partial n}{\partial z} = \frac{\partial n}{\partial m}\frac{\partial m}{\partial z} + \frac{\partial n}{\partial \beta}\frac{\partial \beta}{\partial z},$$

and therefore

$$\begin{aligned}\nu U &= \frac{\partial n}{\partial t} + u\frac{\partial n}{\partial x} + v\frac{\partial n}{\partial y} + w\frac{\partial n}{\partial z}\\ &= \frac{\partial F}{\partial t} + \frac{\partial n}{\partial m}\left\{\frac{\partial m}{\partial t} + u\frac{\partial m}{\partial x} + v\frac{\partial m}{\partial y} + w\frac{\partial m}{\partial z}\right\}\\ &\qquad + \frac{\partial n}{\partial \beta}\left\{\frac{\partial \beta}{\partial t} + u\frac{\partial \beta}{\partial x} + v\frac{\partial \beta}{\partial y} + w\frac{\partial \beta}{\partial z}\right\}\\ &= \frac{\partial F}{\partial t} = \frac{1}{f}\frac{\partial f}{\partial t}.\end{aligned}$$

Thus the surfaces $U=$const., are made up of vortex lines, but not continually of the same set. This is otherwise evident, since we have

$$\nabla^2\xi = U\xi,\ \nabla^2\eta = U\eta,\ \nabla^2\zeta = U\zeta,$$

$$\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z} = 0,$$

and therefore

$$\xi\frac{\partial U}{\partial x} + \eta\frac{\partial U}{\partial y} + \zeta\frac{\partial U}{\partial z} = 0.$$

As a special instance, we may make the assumption $U=0$, which gives

$$\nabla^2\xi = \nabla^2\eta = \nabla^2\zeta = 0,$$

and we have

$$\frac{\partial f}{\partial t} = 0.$$

In this case we shall have $\lambda_0/\lambda = 1$, and thus the law of conservation of vorticity holds, just as in the case of a perfect fluid.

Another possible assumption is

$$\nu U + c^2 = 0,$$

in which case we have

$$\frac{\partial f}{\partial t} + c^2 f = 0,$$

and therefore

$$\frac{\lambda_0}{\lambda} = e^{-c^2t} F(m, \beta),$$

which indicates a gradual decay of the vorticity. A method of securing an example of this last case is by assuming

$$u = \frac{\partial N}{\partial y} - \frac{\partial M}{\partial z}, \quad v = \frac{\partial L}{\partial z} - \frac{\partial N}{\partial x}, \quad w = \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y},$$

where L, M, N are three solutions of the equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} + c^2\phi = 0,$$

connected by the relation

$$\frac{\partial L}{\partial x} + \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z} = 0.$$

ON THE ERRORS OF PRINCIPAL AND SUBSIDIARY CONVERGENTS TO A CONTINUED FRACTION.

By *R. Hargreaves, M.A.*, formerly Fellow of St. John's College, Cambridge.

THE usual limits assigned to the error of any convergent to a continued fraction assume only a knowledge of the next following partial quotient, and for the earlier convergents may be rather wide. In practical applications ease of manipulation is so important, that it is often better to use a simple convergent and apply a correction, than to work with a higher convergent, accurate but unwieldy.

In what follows a method is given for finding the error of a simple convergent, either accurately when the continued fraction is finite, or when infinite with the accuracy of an assigned higher convergent. The process, which I think will be found an economic one, is also applied to the subsidiary or intermediate convergents. Further, a rule is given (not depending on the special process) for deciding whether a subsidiary following a principal convergent, is superior or inferior to it in accuracy. I have not been able to find this rule, but its simplicity makes it rather a matter of surprise, if it has hitherto been overlooked.

§ 1. Let p_n and q_n denote the numerator and denominator of any convergent to the continued fraction F, a_n being the partial and a_n+f_n the complete quotient at this stage, so that

$$f_n=\frac{1}{a_{n+1}+f_{n+1}} \text{ ....(1)},$$

then $F \sim \frac{p_n}{q_n}=\frac{1}{q_n\{(a_{n+1}+f_{n+1})\,q_n+q_{n-1}\}}=\frac{1}{q_n\,(q_{n+1}+f_{n+1}q_n)}$,

the first being the usual form given to the error. Multiplying numerator and denominator by $a_{n+2}+f_{n+2}$, and making use of (1) to simplify the resulting denominator,

$$F \sim \frac{p_n}{q_n}=\frac{a_{n+2}+f_{n+2}}{q_n\{(a_{n+2}+f_{n+2})\,q_{n+1}+q_n\}}=\frac{a_{n+2}+f_{n+2}}{q_n\,(q_{n+2}+f_{n+2}q_{n+1})}.$$

Repeating the process with $a_{n+3}+f_{n+3}$ for multiplier,

$$F \sim \frac{p_n}{q_n}=\frac{(a_{n+3}a_{n+2}+1)+a_{n+2}f_{n+3}}{q_n\,(q_{n+3}+f_{n+3}q_{n+2})}.$$

Now write $\varpi_0=0$, $\varpi_1=1$, $\varpi_2=a_{n+2}$, and generally

$$\varpi_r=a_{n+r}\varpi_{r-1}+\varpi_{r-2} \text{ ....(2)},$$

true down to $r=2$, then

$$F \sim \frac{p_n}{q_n}=\frac{\varpi_r+f_{n+r}\varpi_{r-1}}{q_n\,(q_{n+r}+f_{n+r}q_{n+r-1})} \text{ ....(3)}.$$

In fact

$$\frac{(\varpi_r+f_{n+r}\varpi_{r-1})\,(a_{n+r+1}+f_{n+r+1})}{(q_{n+r}+f_{n+r}q_{n+r-1})\,(a_{n+r+1}+f_{n+r+1})}=\frac{(a_{n+r+1}\varpi_r+\varpi_{r-1})+\varpi_r f_{n+r+1}}{(a_{n+r+1}q_{n+r}+q_{n+r-1})+f_{n+r+1}q_{n+r}}$$

$$=\frac{\varpi_{r+1}+f_{n+r+1}\varpi_r}{q_{n+r+1}+f_{n+r+1}q_{n+r}},$$

in virtue of (1) and (2).

This establishes the induction, and the opening terms obviously fit into the general formula. As a special case, if we write $f_{n+r}=0$ in (3), we have $\frac{p_{n+r}}{q_{n+r}}$ in lieu of F, and accordingly

$$\frac{p_{n+r}}{q_{n+r}} \sim \frac{p_n}{q_n}=\frac{\varpi_r}{q_nq_{n+r}} \text{ ....(4)}.$$

Again, limits to the error at any stage are got by writing $f_{n+r}=0$ and 1 in (3), viz. $\frac{\varpi_r}{q_n q_{n+r}}$ and $\frac{\varpi_r+\varpi_{r-1}}{q_n(q_{n+r}+q_{n+r-1})}$, these taking alternately the places of upper and lower limits. The width of these limits is $\frac{\varpi_r q_{n+r-1}\sim\varpi_{r-1}q_{n+r}}{q_n q_{n+r}(q_{n+r}+q_{n+r-1})}$, and from the sequence formulæ

$$\varpi_r q_{n+r-1}-\varpi_{r-1}q_{n+r}=-(\varpi_{r-1}q_{n+r-2}-\varpi_{r-2}q_{n+r-1}),$$

and therefore by successive steps $=\pm q_n$; hence the width of the limits is

$$\frac{1}{q_{n+r}(q_{n+r}+q_{n+r-1})}\text{....................}(5).$$

But if in $F\sim\frac{p_{n+r-1}}{q_{n+r-1}}=\frac{1}{q_{n+r-1}(q_{n+r}+f_{n+r}q_{n+r-1})}$ we write $f_{n+r}=0$ or 1, the limits given are $\frac{1}{q_{n+r-1}q_{n+r}}$ and $\frac{1}{q_{n+r-1}(q_{n+r}+q_{n+r-1})}$, and the difference of these is $\frac{1}{q_{n+r}(q_{n+r}+q_{n+r-1})}$ agreeing with the above. The necessity for this appears when we consider that a knowledge of a_{n+r} is required to give the limits above for the error of $\frac{p_{n+r-1}}{q_{n+r-1}}$, and this is precisely the extent of knowledge assumed in obtaining (5) as the range of the error in (3).

As a numerical illustration take

$$(1)\ \frac{1}{1}+\frac{1}{2}+\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\frac{1}{2}+\frac{1}{1}+\frac{1}{4}+\frac{1}{2}+\frac{1}{3}$$

of which the convergents are

$$1,\ \frac{2}{3},\ \frac{3}{4},\ \frac{11}{15},\ \frac{58}{79},\ \frac{127}{173},\ \frac{185}{252},\ \frac{867}{1181},\ \frac{1918}{2614},\ \frac{6624}{9023}.$$

Apply (4) to find the difference between $\frac{11}{15}$ and any of the succeeding convergents.

The ϖ's are 0, 1, $\overset{2}{2}$, $\overset{1}{3}$, $\overset{4}{14}$, $\overset{2}{31}$, $\overset{3}{107}$ the partial quotients concerned being written above them as a guide to the formation.

Then $\frac{11}{15} \sim \frac{127}{173} = \frac{2}{15 \times 173}$,

$$\frac{11}{15} \sim \frac{19}{2614} = \frac{31}{15 \times 2614}, \quad \frac{11}{15} \sim \frac{6624}{9023} = \frac{107}{15 \times 9023}.$$

Without the use of this method such a difference requires the formation of the p's, and the multiplications and subtraction involved in $p_{n+r}q_n - p_n q_{n+r}$. These last are dispensed with, and in lieu of the calculation of p's, we have that of ϖ's following exactly the same sequence law as the p's, but starting with 0, 1 instead of p_n and p_{n+1}, and so involving much smaller figures. The exact error $\frac{11}{15} - F$ is given in the successive forms

$$\frac{1}{15(79+15f_5)}, \quad \frac{2+f_6}{15(173+79f_6)}, \quad \frac{3+2f_7}{15(252+173f_7)}, \quad \cdots \quad \frac{107}{15 \times 9023},$$

where $f_5 = \frac{1}{2}\,\frac{}{+}\,\frac{1}{1}\,\frac{}{+}\,\frac{1}{4}\,\frac{}{+}\,\frac{1}{2}\,\frac{}{+}\,\frac{1}{3}$, $f_6 = \frac{1}{1}\,\frac{}{+}\,\frac{1}{4}\,\frac{}{+}\,\frac{1}{2}\,\frac{}{+}\,\frac{1}{3}$, $\ldots f_{10} = 0$.

As an example of the approximative use take

$$\sqrt{(13)} = 3 + \frac{1}{1}\,\frac{}{+}\,\frac{1}{1}\,\frac{}{+}\,\frac{1}{1}\,\frac{}{+}\,\frac{1}{1}\,\frac{}{+}\,\frac{1}{6}\,\frac{}{+}\,\frac{1}{1}\,+ \ldots .$$

The convergents are

$$\frac{3}{1}, \quad \frac{4}{1}, \quad \frac{7}{2}, \quad \frac{11}{3}, \quad \frac{18}{5}, \quad \frac{\ }{33}, \quad \frac{\ }{38}, \quad \frac{\ }{71}, \quad \frac{\ }{109}, \quad \frac{\ }{180}, \quad \frac{\ }{1189} \ldots.$$

Wishing to use $\frac{18}{5}$ as convergent, we compare its error with that of the 10th just preceding the second 6, which itself has an error roughly $\frac{1}{200{,}000}$. The series of ϖ's is

$$0, \ 1, \ \overset{1}{1}, \ \overset{1}{2}, \ \overset{1}{3}, \ \overset{1}{5} \ \text{and} \ \frac{(\ )}{180} - \frac{18}{5} = \frac{5}{5 \times 180} = \frac{1}{180}.$$

The result $3\frac{3}{5} + \frac{1}{180}$ is correct to five places, easy to apply, and readily got at.

§ 2. A slight modification is wanted to adapt the process to a subsidiary convergent. When limits are assigned to the denominator of a convergent, and it is desired to find the

closest in excess and defect, of the two that which has the smaller denominator is always a principal convergent, the other may be a subsidiary. Let $\frac{p_n}{q_n}$ be the principal convergent, $\frac{p'_{n+1}}{q'_{n+1}}$ the subsidiary, differing from $\frac{p_{n+1}}{q_{n+1}}$ merely in being formed with a smaller partial quotient than a_{n+1}, say a'_{n+1}. We may write $a_{n+1} = a'_{n+1} + a''_{n+1}$, and a'_{n+1} may have all integral values from 1 to $a_{n+1} - 1$. Thus

$$F \sim \frac{p'_{n+1}}{q'_{n+1}} = \frac{(a_{n+1} + f_{n+1})\, p_n + p_{n-1}}{(a_{n+1} + f_{n+1})\, q_n + q_{n-1}} \sim \frac{a'_{n+1} p_n + p_{n-1}}{a'_{n+1} q_n + q_{n-1}}$$

$$= \frac{a''_{n+1} + f_{n+1}}{q'_{n+1}\,(q_{n+1} + f_{n+1} q_n)} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

The process applied to prove (3) now gives generally

$$F \sim \frac{p'_{n+1}}{q'_{n+1}} = \frac{\varpi'_r + f_{n+r}\varpi'_{r-1}}{q'_{n+1}\,(q_{n+r} + f_{n+r} q_{n+r-1})} \;\ldots\ldots\ldots\ldots (7),$$

where $\varpi'_r = a_{n+r}\varpi'_{r-1} + \varpi'_{r-2}$ as for ϖ_r, but $\varpi'_0 = 1$, $\varpi'_1 = a''_{n+1}$, so that, in fact, $\frac{\varpi'_r}{\varpi_r}$ is a convergent to the value of

$$a''_{n+1} + \frac{1}{a_{n+2}} + \frac{1}{a_{n+3} + \ldots}\,.$$

As a particular case

$$\frac{p_{n+r}}{q_{n+r}} \sim \frac{p'_{n+1}}{q'_{n+1}} = \frac{\varpi'_r}{q'_{n+1} q_{n+r}} \;\ldots\ldots\ldots\ldots\ldots (8).$$

The limits for the error in (7) differ from those written for (3) only by having ϖ'_r for ϖ_r and q'_{n+1} for q_n, and the errors are on opposite sides of F.

As numerical example, in (1) take the third subsidiary after the fourth principal convergent $\frac{11}{15}$, viz. $\frac{36}{49}$ formed thus $36 = 3 \times 11 + 3$, $49 = 3 \times 15 + 4$. Here $a_5'' = 5 - 3 = 2$, and

$\quad\;\; 2 \;\; 2 \;\; 1 \;\; 4 \;\; 2 \;\; 3$

the ϖ''s are 1, 2, 5, 7, 33, 73, 252. Thus by (8)

$$\frac{36}{49} \sim \frac{185}{252} = \frac{7}{49 \times 252}, \quad \frac{36}{49} \sim \frac{6624}{9023} = \frac{252}{49 \times 9023},$$

and the exact error has the forms

$$\frac{2 + f_5}{49\,(79 + 15 f_5)}, \; \frac{5 + 2 f_6}{49\,(173 + 79 f_6)}, \; \ldots, \; \frac{252}{49 \times 9023},$$

where the f's have the same meaning as before. Again, take the middle subsidiary formed after the seventh convergent, viz. $\frac{497}{677}$. To find the difference between $\frac{36}{49}$ and $\frac{497}{677}$, two subsidiaries in different places, use 2 instead of 4 as seventh partial quotient, and so $\varpi_4' = 2 \times 7 + 5 = 19$ instead of 33 belonging to the next principal convergent, and $\frac{36}{49} - \frac{497}{677} = \frac{19}{49 \times 677}$.

§ 3. For a criterion to determine whether $\frac{p_n}{q_n}$ or $\frac{p'_{n+1}}{q'_{n+1}}$ is the closer to F, we do not require the general forms (3) and (7), the usual case for which $r = 1$ is sufficient. Discarding the bracket-factor in the denominator, which is alike for both, $\frac{p_n}{q_n}$ is a closer convergent than $\frac{p'_{n+1}}{q'_{n+1}}$, if

$$\frac{a''_{n+1} + f_{n+1}}{q'_{n+1}} > \frac{1}{q_n},$$

or if

$$a''_{n+1} + f_{n+1} > \frac{q'_{n+1}}{q_n},$$

or if

$$a''_{n+1} + \frac{1}{a_{n+2} +} \frac{1}{a_{n+3} +} \dots > a'_{n+1} + \frac{1}{a_n +} \frac{1}{a_{n-1} +} \dots + \frac{1}{a_1} \dots (9).$$

When a_{n+1} is odd the integral part decides the question, the number of subsidiaries is even, the first half being worse, the second half better, convergents than $\frac{p_n}{q_n}$. When a_n is even, the integral part leaves the case of the middle subsidiary an open question, which is decided by the *first* of the pairs (a_{n+2}, a_n), (a_{n+3}, a_{n-1}), ... shewing a difference; if in an odd place in (9) this figure on the left is greater, or in an even place less, than that on the right, then $\frac{p_n}{q_n}$ is the better convergent, when the inequalities are reversed the worse. If with a_{n+1} even the continued fraction terminates with a_{n+1}, the middle subsidiary is the better convergent, the right-hand member of (9) having a fractional part, the left none. It is also clear that if a_{n+1} is even, any symmetry in the partial quotients before and after, by deferring the point of difference in (9), makes the errors of the principal convergent and middle subsidiary almost identical, and the mean of the two convergents an exceptionally close approximation.

Noting that the subsidiary is always to the right of the principal convergent in question, the criterion may be easily remembered thus:—Error of right is less if the continued fraction to right is less, and and *vice-versâ*. Thus in $\frac{1}{2}+\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{5}+\frac{1}{4}$ comparing the third principal and middle subsidiary following, the subsidiary is better because $3+\frac{1}{5+\dots}<3+\frac{1}{4+\dots}$. But in $\frac{1}{2}+\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{4}+\frac{1}{4}$ the subsidiary is worse because $3+\frac{1}{4}+\frac{1}{4}>3+\frac{1}{4}+\frac{1}{2}+\frac{1}{2}$. In the first of these examples comparing the fourth principal with the subsidiaries following, four in number; the first two are definitely worse, the last two definitely better than the principal, the integral part deciding it, *e.g.* for the third of these subsidiaries it turns on $2+\frac{1}{4}$ being less than $3+\frac{1}{6+\dots}$.

A good example is also furnished by $\sqrt{2}=1+\frac{1}{2}+\frac{1}{2+\dots}$ of which the sixth principal convergent is $\frac{99}{70}$ and the subsidiary following $\frac{140}{99}$. To decide which is the better, we have to compare $1+\frac{1}{2}+\frac{1}{2+\dots}$ with $1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}$ terminating.

The former is the less as 2 its seventh quotient (odd place) is less than ∞ the seventh quotient of the latter, and accordingly $\frac{140}{99}$ is very slightly the better convergent. Each of these approximations is easy to calculate, and the error of the mean is in defect by ·00000000194....

The continued fraction $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{3}+\frac{1}{2}$ is constructed to make the errors of the third convergent, and the middle subsidiary following exactly equal (apply (9)), and the whole fraction is therefore the mean of these convergents.

The same is true generally of

$$a_0+\frac{1}{a_1}+\dots\frac{1}{a_n}+\frac{1}{2a_{n+1}}+\frac{1}{a_n}+\dots+\frac{1}{a_1},$$

where $2a_{n+1}$ is written to shew plainly that the quotient is even. The direct proof of this forms an interesting exercise, which may be left to the reader.

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ON THE OSCILLATIONS OF A HETEROGENEOUS COMPRESSIBLE LIQUID SPHERE AND THE GENESIS OF THE MOON; AND ON THE FIGURE OF THE MOON.

By *W. F. Sedgwick, M.A.*, Trinity College, Cambridge.

11

I. On the oscillations of a heterogeneous compressible liquid sphere and the genesis of the moon.

In his paper "On the Precession of a Viscous Spheroid, and on the Remote History of the Earth" (*Phil. Trans.* 1879, Part II., p. 537, etc.), Prof. Darwin has traced back the history of the moon to a period at which it and the earth were rotating round one another, as if forming parts of one rigid body, in a period of some 5 or 5½ hours. In a later paper on "The Secular changes in the Elements of the Orbit of a Satellite, &c." (*Phil. Trans.* 1880, Part II., pp. 834–836) he gave reasons for modifying this period to something between 2 and 2½ hours possibly, but adds in an appendix that 'it is not possible to make an adequate consideration of the subject......without a treatment of the theory of the tidal friction of a planet attended by a pair of satellites.' Mr. Love ("On the Oscillations of a Rotating Liquid Spheroid and the Genesis of the Moon," *Phil. Mag.*, March, 1889, Vol. XXVII., p. 255) considers a period between 2 and 4 hours as most probable. The same author quotes the researches of Riemann and Poincaré on the stability of a rotating liquid ellipsoid as proving that 'when the density' of a rotating spheroid 'is not less than 3, and the period of rotation longer than 3 hours,* the motion is certainly stable.'

Now Prof. Darwin considers it highly probable that the earth had contracted to nearly its present dimensions before the genesis of a satellite (*Phil. Trans.* II. 1881, p. 532 and § 9 generally). Hence we may take it that the density at this period was greater than 3. If at the same time the period of rotation was greater than three hours, the question arises as to how the instability can have set in which caused the genesis of the moon (for we seem driven to suppose that the moon

* More strictly 2⅜ hours.

did once form part of the earth). Prof. Darwin (in the paper first quoted) suggested that the instability may have been due to coincidence in period between the semi-diurnal tide and the free oscillation, and quoted Sir W. Thomson's result for the period of a homogeneous liquid sphere of the same mean density as the earth, and supposed to be oscillating ellipsoidally, viz. $1^h\ 34^m$. This result not agreeing very well with the half period of rotation then obtained by him, he suggested that if the sphere were supposed heterogeneous, Sir W. Thomson's period would probably be increased.

Mr. Love has since taken up the subject (in the paper referred to above), and has obtained the periods of oscillation of a rotating liquid spheroid of the same nature as the earth at the period in question. His results shew that 'for a liquid spheroid of the same mean density as the earth, the longest period is always very nearly equal to $1\frac{1}{2}$ hours......whatever the rate of rotation may be' (pp. 255, 256; *cf.* the table on p. 264). This period, it will be observed, is very nearly the same as Sir W. Thomson's, and indicates that for a comparatively slight departure from spherical form, such as is the case with the spheroids in question, the period is very nearly the same as that of the non-rotating sphere with the same mean density, as would otherwise be expected.

The like approximate equality in period will almost certainly hold good if we suppose the spheroids and sphere to be heterogeneous, provided the variation of density as we recede from the centre is the same in both. Hence, in accordance with the suggestion of Prof. Darwin referred to above, I thought that it would be interesting, in connection with the present hypothesis as to the mode of genesis of the moon, to investigate the period (or periods) of oscillation of a non-rotating heterogeneous liquid sphere. The most usual law of density to assume for the earth is Laplace's, viz. $\rho \propto \dfrac{\sin nr}{r}$, but with this law the equations seemed to defy integration. Prof. Darwin (*Proc. Roy. Soc.*, 1883, p. 158) has suggested another law, viz. $\rho = \dfrac{C}{r}$, where C is a constant, which represents the known facts almost equally well with Laplace's (cf. Osmond Fisher, *Physics of the Earth's Crust*, p. 35, &c.). It gives indeed infinite density at the centre, but this will hardly be a serious objection in our problem, for it will be seen from the form of the velocity potential that there is no motion at the centre, so that the infinite density at this point does not really concern the hydrodynamical problem.

Moreover, the mass concentrated in a small sphere surrounding the centre tends to zero as the radius of the sphere diminishes indefinitely; and our results will, in fact, differ very little from those which would be obtained on the supposition of a small rigid spherical nucleus at the centre, of large, but finite density.

Consider first the conditions holding in the equilibrium of the heterogeneous liquid sphere.

The law of density is

$$\rho = \frac{C}{r},$$

where C is a constant and r is the distance from the centre.

The equilibrium gravitation potential V is

$$V = \frac{\gamma}{r}\int_0^r 4\pi r^2 dr \frac{C}{r} + \gamma \int_0^a 4\pi r\, dr \frac{C}{r}$$
$$= 2\pi\gamma C(2a - r),$$

where γ is the constant of gravitation, a the radius of the sphere.

Also $$dp = \rho dV,$$

where p is the hydrostatic pressure at any point, and therefore

$$dp = -2\pi\gamma C\rho\, dr = 2\pi\gamma C^2 \frac{d\rho}{\rho} \quad \text{..............(1).}$$

This equation gives the law connecting the pressure and density in equilibrium. It gives the law of compressibility of the fluid, and hence will hold also in small oscillations if we may assume the small increase of density to take place instantaneously on the application of a small increase of pressure, as will practically be the case. Or, more strictly, this relation (1) will hold, both in equilibrium and small motions, throughout any portion of the liquid consisting of the same chemical substance, and since we may suppose the different substances of which the whole sphere is made up to be stratified in layers, across the boundaries between which the pressure and density will be continuous, it may also be said to hold throughout the whole sphere.

Now consider the problem of small motions. The method adopted is similar to that of Sir W. Thomson for the homogeneous sphere (*Mathematical and Physical Papers*, III.,

pp. 384–386). Since p is a function of ρ, a velocity-potential ϕ exists (Lamb, *Hydrodynamics*, § 18). Hence, if u, v, w be the velocities in the directions of x, y, and z,

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}, \quad w = \frac{\partial \phi}{\partial z}.$$

For small motions

$$-\frac{\partial p}{\partial x} = \rho \left(\frac{\partial u}{\partial t} - \frac{\partial V}{\partial x} \right), \text{ \&c.};$$

therefore from (1)

$$-2\pi\gamma C^2 \frac{1}{\rho^2} \frac{\partial \rho}{\partial x} = \frac{\partial u}{\partial t} - \frac{\partial V}{\partial x}, \text{ \&c.};$$

whence, on integrating,

$$\text{constant} + 2\pi\gamma C^2 \frac{1}{\rho} = \frac{\partial \phi}{\partial t} - V \;\ldots\ldots\ldots\ldots(2),$$

the part of the constant which involves t being supposed, as usual, to be included in ϕ, which will not alter the velocities $\frac{\partial \phi}{\partial x}$, &c.

The equation of continuity may be written

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} \frac{\partial \phi}{\partial x} + \ldots + \rho \nabla^2 \phi = 0,$$

or
$$\frac{\partial \rho}{\partial t} - \frac{C}{r^3} \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right) \phi + \frac{C}{r} \rho \nabla^2 \phi = 0 \;\ldots.(3).$$

Let h = radial component of displacement at surface $r = a$. Then

$$h = S_1 + S_2 + \ldots + S_i + \ldots,$$

where S_i is a spherical surface harmonic of order i.

Since $\frac{dh}{dt}$ is the radial component of velocity, we have

$$x \frac{\partial \phi}{\partial x} + y \frac{\partial \phi}{\partial y} + z \frac{\partial \phi}{\partial z} = a \Sigma \frac{dS_i}{dt}, \text{ when } r = a \;\ldots\ldots(4).$$

Also
$$p = g\rho_0 h + \pi, \text{ when } r = a \;\ldots\ldots(5),$$

g being the surface value of gravity, ρ_0 the surface density, and π the external pressure.

Differentiate (5) with respect to t, put $h=\Sigma S_i$, and substitute from (1) and (4), and, having thus eliminated ΣS_i, we obtain the single surface condition

$$\frac{1}{\rho}\frac{\partial\rho}{\partial t}=\frac{1}{a^2}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)\phi, \text{ when } r=a\ldots(6),$$

since $g=2\pi\gamma C$ (= same at all depths); and we write $\frac{\partial\rho}{\partial t}$, and not $\frac{d\rho}{dt}$, since the variation takes place at a fixed point on the fixed spherical surface $r=a$.

Now we may put

$$\phi=\Sigma\,\phi(r)\left(\frac{r}{a}\right)^i\Phi_i,$$

where $\phi(r)$ is a function of r, and Φ_i is a spherical surface harmonic of order i. The equations being linear in the Φ's, we need, as usual, consider only one of them. Equation (3) becomes

$$\frac{1}{C}\frac{\partial\rho}{\partial t}=-\left\{\frac{1}{r}\phi''(r)+\frac{2i+1}{r^2}\phi'(r)-\frac{i}{r^3}\phi(r)\right\}\left(\frac{r}{a}\right)^i\Phi_i\ldots(7).$$

Also equation (2) may be written, on differentiation with regard to t,

$$-2\pi\gamma C^2\frac{1}{\rho^2}\frac{\partial\rho}{\partial t}=\frac{\partial^2\phi}{\partial t^2}-\frac{\partial V}{\partial t}\ldots\ldots\ldots\ldots\ldots(8).$$

Now $\frac{\partial V}{\partial t}$ consists of two parts, (1) that depending on the small variation of density within the mean sphere $r=a$, and (2) that depending on the harmonic inequality at the surface. Whence we have, by (7) and the known formulæ (Thomson and Tait, § 542),

$$\frac{\partial V}{\partial t}=-\frac{4\pi\gamma C}{2i+1}\left[\frac{1}{a^i r^{i+1}}\int_0^r\left\{r^{2i+1}\phi''(r)+(2i+1)r^{2i}\phi'(r)-i\,r^{2i-1}\phi(r)\right\}dr\right.$$
$$\left.+\left(\frac{r}{a}\right)^i\int_r^a\left\{\phi''(r)+\frac{2i+1}{r}\phi'(r)-\frac{i}{r^2}\phi(r)\right\}dr\right]\Phi_i$$
$$+\frac{4\pi\gamma C}{2i+1}\left(\frac{r}{a}\right)^i\frac{dS_i}{dt};$$

equation (8) may therefore be written

$$-\left[r\phi''(r)+(2i+1)\phi'(r)-\frac{i}{r}\phi(r)\right]\left(\frac{r}{a}\right)^i\Phi_i$$

$$+\phi(r)\left(\frac{r}{a}\right)^i\frac{1}{2\pi\gamma C}\frac{d^2\Phi_i}{dt^2}-\frac{2}{2i+1}\left(\frac{r}{a}\right)^i\frac{dS_i}{dt}$$

$$+\frac{2}{2i+1}\left[-\frac{i}{r^{2i+1}}\int_0^r r^{2i-1}\phi(r)\,dr+(i+1)\int_r^a\frac{1}{r}\phi'(r)\,dr\right.$$

$$\left.+\phi'(a)+\frac{i}{a}\phi(a)-\frac{i}{r}\phi(r)\right]\left(\frac{r}{a}\right)^i\Phi_i=0\ \ldots\ldots(9).$$

Divide by $\left(\frac{r}{a}\right)^i$ and differentiate with regard to r, multiply the result by $r^{2(i+1)}$, differentiate with regard to r again, and finally divide throughout by r^{2i-1}, and we obtain the equation

$$-\left[r^4\phi^{iv}(r)+(4i+5)r^3\phi'''(r)\right.$$

$$\left.+\{4(i+1)^2-i\}r^2\phi''(r)-2i^2r\phi'(r)+2i^2\phi(r)\right]\Phi_i$$

$$+2\left[-r^2\phi''(r)-(2i+1)r\phi'(r)+i\phi(r)\right]\Phi_i$$

$$+\left[r^3\phi''(r)+2(i+1)r^2\phi'(r)\right]\frac{1}{2\pi\gamma C}\frac{d^2\Phi_i}{dt^2}=0.$$

Now we may expand Φ_i in a series of sines and cosines of multiples of t, for each of which independently the above equation, being true for all time, must hold good.

Hence, if for a typical term

$$\frac{d^2\Phi_i}{dt^2}=-p^2\Phi_i,$$

and we write $$\lambda=\frac{p^2}{2\pi\gamma C}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10),$$

the equation becomes

$$r^4\phi^{iv}(r)+(4i+5)r^3\phi'''(r)+\{4(i+1)^2-(i-2)\}r^2\phi''(r)$$

$$-2(i^2-2i-1)r\phi'(r)+2i(i-1)\phi(r)$$

$$+\lambda r\{r^2\phi''(r)+2(i+1)r\phi'(r)\}=0\ \ldots\ldots\ldots(11),$$

in which λr is of zero dimensions.

We may obtain a solution in the form of a sum of four series as follows:

$$\phi(r) = A\{r^{m_1} + \lambda a_1 r^{m_1+1} + \lambda^2 a_2 r^{m_1+2} + \dots$$
$$+ B\{r^{m_2} + \lambda b_1 r^{m_2+1} + \lambda^2 b_2 r^{m_2+2} + \dots$$
$$+ C\{r^{m_3} + \lambda c_1 r^{m_3+1} + \lambda^2 c_2 r^{m_3+2} + \dots$$
$$+ D\{r^{m_4} + \lambda d_1 r^{m_4+1} + \lambda^2 d_2 r^{m_4+2} + \dots\dots\dots \quad (12).$$

A, B, C, D are the four constants of the solution. [There will be no occasion to confuse the C here used with that in the expression for the density, as it will be presently shewn that we must take $C = 0 = D$.]

m_1, m_2, m_3, m_4 are the roots of

$$m(m-1)(m-2)(m-3) + (4i+5)m(m-1)(m-2)$$
$$+ (4i^2+7i+6)m(m-1) - 2(i^2-2i-1)m + 2i(i-1) = 0,$$

or

$$f(m) \equiv \{m^2 + (2i-1)m - 2(i-1)\}\{m^2 + 2im - i\} = 0 \dots (13).$$

Writing $$\psi(m) \equiv m^2 + (2i+1)m \dots\dots\dots\dots\dots (14),$$

we find

$$a_1 = -\frac{\psi(m_1)}{f(m_1+1)}, \quad a_2 = \frac{\psi(m_1)\,\psi(m_1+1)}{f(m_1+1)\,f(m_1+2)},$$

$$a_3 = -\frac{\psi(m_1)\,\psi(m_1+1)\,\psi(m_1+2)}{f(m_1+1)\,f(m_1+2)\,f(m_1+3)}, \text{ \&c. } \dots\dots (15),$$

with similar expressions for the b, c, d coefficients in terms of m_2, m_3, m_4 respectively.

Also, arranging m_1, m_2, m_3, m_4 in descending order of magnitude, we have, by solution of (13),

$$m_1 = -i + \tfrac{1}{2} + \sqrt{(i^2 + i - \tfrac{7}{4})}, \quad m_2 = -i + \sqrt{(i^2 + i)},$$
$$m_3 = -i + \tfrac{1}{2} - \sqrt{(i^2 + i - \tfrac{7}{4})}, \quad m_4 = -i - \sqrt{(i^2 + i)} \quad \dots (16).$$

Hence no two of the quantities m differ by a whole number, and the four series in (12) are therefore distinct. They are also convergent. For, to take the series corresponding to m_1, the ratio of the $(n+1)^{\text{th}}$ term to the n^{th} is

$$\lambda r \frac{a_n}{a_{n-1}} = -\lambda r \frac{\psi(m_1+n-1)}{f(m_1+n)}$$

by (15), which by (13) and (14) is zero in the limit when n is infinite; and similarly for the remaining series. It follows also that the series are absolutely convergent. Thus (12) is the complete solution of (11).

For all values of $i > 1$, m_3 and m_4 are negative, and numerically $> i$. The velocity potential is of the order $\phi(r)\, r^i$ in r, hence, provided $i \gtreqless 2$, we see from (12) that C and D must be zero, or there will be infinite velocity at the centre.

The ratio $A : B$ must be so chosen as to satisfy (9) and the boundary condition (6). By (4) we have

$$\frac{dS_i}{dt} = \frac{1}{a}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\phi(r)\left(\frac{r}{a}\right)^i \Phi_i, \text{ when } r = a,$$

$$= \left\{\phi'(a) + \frac{i}{a}\phi(a)\right\}\Phi_i.$$

Since also by (10),

$$\frac{d^2\Phi_i}{dt^2}\,\frac{1}{2\pi\gamma C} = -\lambda\Phi_i,$$

equation (9) may be written in the form

$$r^2\phi''(r) + (2i+1)\,r\phi'(r) - \frac{i(2i-1)}{2i+1}\,\phi(r) + \lambda r\phi(r)$$

$$+\frac{2i}{2i+1}\,\frac{1}{r^{2i}}\int_0^r r^{2i-1}\phi(r)\,dr - \frac{2(i+1)}{2i+1}\,r\int_r^a \frac{1}{r}\,\phi'(r)\,dr = 0 \;\ldots(17),$$

the surface harmonic Φ_i cancelling throughout.

By means of (12) to (15) we easily verify that the coefficients of any power of r in this equation vanish, except that multiplying r. In order that the latter may vanish, we must have

$$Aa^{m_1}\left\{\frac{m_1}{m_1-1} + \lambda a_1\frac{m_1+1}{m_1}a + \lambda^2 a_2\frac{m_1+2}{m_1+1}a^2 + \ldots\right\}$$

$$+Ba^{m_2}\left\{\frac{m_2}{m_2-1} + \lambda b_1\frac{m_2+1}{m_2}a + \lambda^2 b_2\frac{m_2+2}{m_2+1}a^2 + \ldots\right\} = 0 \ldots(18).$$

Substituting in (6) from (7), we find

$$a^2\phi''(a) + 2(i+1)\,a\phi'(a) = 0,$$

which gives

$$Aa^{m_1}\{m_1(m_1+2i+1)+\lambda a_1(m_1+1)(m_1+2i+2)a$$
$$+\lambda^2 a_2(m_1+2)(m_1+2i+3)a^2+\dots\}$$
$$+Ba^{m_2}\{m_2(m_2+2i+1)+\lambda b_1(m_2+1)(m_2+2i+2)a$$
$$+\lambda^2 b_2(m_2+2)(m_2+2i+3)a^2+\dots\}=0\dots(19).$$

Just as it was shewn that the series in (12) were absolutely convergent, it may be shewn here that the series in (18) and (19) are absolutely convergent, the ratios of the new factors introduced in (18) and (19) in successive terms being evidently, in the limit, unity. Hence we are justified in eliminating A and B from (18) and (19) by means of the multiplication of the series involved. Accordingly we obtain the period equation

$$\left\{\frac{m_1}{m_1-1}+\lambda a_1\frac{m_1+1}{m_1}a+\dots\right\}$$
$$\times\{m_2(m_2+2i+1)+\lambda b_1(m_2+1)(m_2+2i+2)a+\dots\}$$
$$-\left\{\frac{m_2}{m_2-1}+\lambda b_1\frac{m_2+1}{m_2}a+\dots\right\}$$
$$\times\{m_1(m_1+2i+1)+\lambda a_1(m_1+1)(m_1+2i+2)a+\dots\}=0\dots(20).$$

In this equation the coefficient of $\lambda^r a^r a_s b_{r-s}$ is

$$C_{s,r-s}=\frac{(m_1+s)(m_2+r-s)(m_2-m_1+r-2s)(m_2+m_1+2i+r)}{(m_1+s-1)(m_2+r-s-1)}$$
$$\dots\dots\dots\dots\dots(21).$$

The period equation may be written

$$C_0+C_1\lambda a+C_2\lambda^2a^2+C_3\lambda^3a^3+\dots=0\dots\dots\dots\dots(22),$$

where

$$\left.\begin{aligned}C_0&=C_{0,0},\\ C_1&=a_1C_{1,0}+b_1C_{0,1},\\ C_2&=a_2C_{2,0}+a_1b_1C_{1,1}+b_2C_{0,2}\\ C_3&=a_3C_{3,0}+a_2b_1C_{2,1}+a_1b_2C_{1,2}+b_3C_{0,3}\\ &\&c.\end{aligned}\right\}\dots\dots(23).$$

The a's and b's may be obtained from (13), (14), and (15) by Taylor's expansion. Thus

$$\psi(m+x)=\psi(m)+x\psi'(m)+\frac{x^2}{2}\psi''(m);$$

and the factors of $f(m+x)$ may be found in a similar manner.

The most important case is the one which we have to consider, viz. $i=2$.

After inserting the values of m_1, m_2 in (21), and evaluating the various quantities involved in (23) for the case when $i=2$, we find

$$\left.\begin{aligned} C_0 &= -\ \cdot58724 \\ C_1 &= +\ \cdot67314 \\ C_2 &= -\ \cdot11738 \\ C_3 &= +\ \cdot00922 \\ C_4 &= -\ \cdot00042 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(24),$$

which are intended to be correct to the last figure given.

All the calculations necessary to obtain these values have been made twice independently from the beginning, with almost exactly the same results. One of the two independent series of calculations for the a, b coefficients has also been checked step by step, whilst the accuracy of the $C_{s,r-s}$ coefficients has been verified by putting $\lambda a=1$ in (20) and (22), and comparing the results.

The coefficients C_n decrease with increasing rapidity as n increases. Thus C_5 will only just, if at all, affect the fifth place of decimals, whilst to this order the rest of the coefficients may be neglected altogether. We may approximately obtain any value of x which does not exceed more than two or three units by means of the equation

$$\cdot00042x^4-\cdot00922x^3+\cdot11738x^2-\cdot67314x+\cdot58724=0\ldots(25).$$

This equation has one root between 1 and 2, a second between 8 and 9, and two imaginary roots. The latter are of such a nature that they could not in any case be obtained even approximately by (25) alone, whilst the complete equation should not have any imaginary roots at all. The larger of the real roots cannot be approximated to further by (25) alone, and moreover gives a period of oscillation which is too small to be of any interest in the present connection. Hence we are left with the smaller real root only, which is the only root of the complete equation not exceeding seven or eight units, and gives the longest possible period for ellipsoidal vibrations.

By approximation this root is found to be $x = 1\cdot0493$, where the error may be taken to be less than $\cdot0002$.

Thus we have

$$\lambda a = 1\cdot0493 \quad \text{....................(26)},$$

and from (10),

$$\lambda a = \frac{2\pi}{\gamma}\frac{a}{C}\frac{1}{\tau^2} \quad \text{....................(27)},$$

where τ is the longest possible complete period of oscillation in the mode in question.

Let Δ be the mean density of the earth. Then, using Boys' values for γ and Δ (*Proc. Roy. Soc.*, LVI., p. 132), we have

$$\left.\begin{array}{l}\gamma = 6\cdot6576 \times 10^{-8} \\ \Delta = 5\cdot5270\end{array}\right\} \quad \text{..................(28)}.$$

Also

$$\text{mass of earth} = \tfrac{4}{3}\pi\Delta a^3 = 4\pi\int_0^a \frac{C}{r} r^2\,dr = 2\pi\frac{C}{a}a^3,$$

therefore

$$\frac{C}{a} = \tfrac{2}{3}\Delta = 3\cdot6847 \quad \text{....................(29)}.$$

Substituting in (27) from (26), (28), and (29), we find

$$\tau^2 = \frac{2 \times 3\cdot14159 \times 10^8}{6\cdot6576 \times 3\cdot6847 \times 1\cdot0493},$$

giving

$$\tau = 4941''$$
$$= 1^{\text{hr}}22'21'' \quad \text{....................(30)}$$

accurately to within a second.

The double of the period (30) is very nearly $2\frac{3}{4}$ hours, which lies near the probable limits assigned by Prof. Darwin for the original period of rotation of the earth-moon system. In view of what has been said above as to the magnitude of the roots of equation (22) other than the one here considered, it is unnecessary to consider any smaller periods of oscillation which may exist. For from a comparison of the tables given by Thomson and Tait, § 772, and Lamb, *Hydrodynamics*, pp. 584, 587, it appears that neither Jacobi's nor Maclaurin's ellipsoids exist when the period of rotation < about 8718″ or $2^{\text{hr}}\,25^{\text{m}}$ nearly, at any rate for a homogeneous earth; and the limit will probably not be much altered for a heterogeneous earth. For periods of rotation less than the limit quoted we are not justified in assuming a sphere to be an approximation to the form of the body, and therefore the period (30) is the only one which concerns the present problem. It will be noticed that this period does not differ much from the corre-

sponding period (94′) for the homogeneous sphere with the same mean density (that of the earth), though, contrary to expectation, it is somewhat less, instead of being greater.

With regard to the assumptions made it may be noted that the law of density adopted agrees well with the known facts, and also, at least for some two-thirds of the distance below the surface, gives numerical results not very different from those given by the law of Laplace, which itself also agrees well with the known facts, so that it does not seem probable that the actual distribution of density will differ much from that given by the law adopted. It has already been remarked that the modification to be made on account of rotation of the sphere will probably be small. Perhaps the weakest point lies in the law of compressibility, which certainly appears contrary to our experience. It does not seem easy to estimate the amount of influence the compressibility has on the period of the heterogeneous sphere considered in this paper. I have found that in one or two cases of a sphere composed of two incompressible liquids of considerably different densities, the thickness of the outer shell of lighter liquid being also considerable in comparison with the radius of the whole sphere, the periods (two in number in each case) differed only slightly from the period of the homogeneous incompressible liquid sphere with the same mean density (that of the earth). So far as they go, these results, taken in conjunction with the result of this paper, would appear to indicate that neither the heterogeneity nor the compressibility make much difference to the period, but that the latter depends mainly on the mean density. On the whole we may perhaps consider the period (30) to be probably not very far from the truth.

The result we have obtained seems to lend some additional weight to the suggested hypothesis as to the genesis of the moon, if at least the initial period of rotation should prove to be not much greater than that for which the rotating spheroid first begins to be stable in itself (*cf.* the result quoted from the researches of Riemann and Poincaré on p. 159, and footnote). The question seems to have reached a stage at which it can only be decided by means of a fairly accurate knowledge of what was the initial period of the earth-moon, and this we hardly possess as yet. For the present therefore we must be content to leave the matter as it stands.

It seems worthy of note that, if we suppose a satellite to revolve in a circular orbit round a primary obeying the assumed law of density, then, when the radius of the primary

is ·64 times that of the satellite's orbit, the period of the tide produced in the primary by the satellite will coincide with the free period of vibration of the primary. Instability therefore ensues, and it is very possible that it would cease only by the separation of the primary into two bodies. On the assumption then of a contracting spherical central body obeying a law of density somewhat of the nature assumed (and even a considerable divergence from this law would probably not produce much difference in the natural period of vibration), it appears that we might account for the evolution of a system not very different in its general aspects from the solar system. Thus each planet or satellite would be supposed in its turn to evolve a new one. The initial satellite in each subsystem might be produced in the same manner as the moon on the hypothesis suggested, whilst the initial planet of the system might be caused by a similar, or different, agency in long distant periods. Unfortunately, however, the ratio ·64 is appreciably greater than that indicated by Bode's law, whilst the theory of tidal evolution only provides for a slight increasing of the distance from the sun. In the solar sub-systems, on the other hand, we find some cases where the ratio of the distances of successive satellites from their primary is somewhat greater than ·64. Whether or not the principle in question ever had anything to do with planetary evolution, it seems curious that the ratio ·64 should fall comparatively so near to the corresponding ratios actually observed in our system.

II. On the figure of the moon.

In Routh's *Rigid Dynamics* (Vol. II., 1892, § 573) a difficulty is noticed in connection with the figure of the moon (*cf.* also Tisserand, *Mécanique Céleste*, II., ch. XXVIII., Art. 204). If A, B, C be the moments of inertia of the moon about its principal axes through the centre of gravity, the theoretical values as calculated by Laplace are

$$\frac{B-A}{C} = \cdot 00000\ 03618\lambda, \quad \frac{C-A}{C} = \cdot 00000\ 04824\lambda,$$

where $\lambda =$ ratio of mass of earth to mass of moon. If we put $\lambda = 83$, we have

$$\frac{B-A}{C} = \cdot 00003\ 00, \quad \frac{C-A}{C} = \cdot 00004\ 00, \quad \frac{C-B}{C} = \cdot 00001\ 00 \quad \ldots\ldots\ldots\ldots(1)$$

(the third ratio being obtained by subtraction of the other two).

The best results of observation are apparently those obtained at Königsberg, viz.

$$\frac{B-A}{C} = \cdot 00031\,5,\ \frac{C-A}{C} = \cdot 00061\,4,\ \frac{C-B}{C} = \cdot 00029\,9 \ldots (2).$$

The ratios of the numbers (2) to (1) are respectively

$$10\cdot 5,\ 15\cdot 3,\ 29\cdot 9 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

which should of course reduce to unity, if the theory were correct.

Laplace's results were obtained on the hypothesis that the moon solidified at its present distance from the earth. But we now know, as the result of Prof. Darwin's researches, that the moon has in all probability receded to its present distance from a period when it was almost in contact with the earth. The question therefore arises whether the moon may not have solidified at a much earlier period than one corresponding to its present distance from the earth, and, if so, whether we cannot make the theoretical and observed values agree much more closely than those above given. The moon being a comparatively small body, and at present containing apparently very little of its original heat, it is *a priori* probable that it cooled at an early period in its history.

It is one of the results of Prof. Darwin's researches that the moon has constantly turned the same face to the earth since a very early period, in fact before the month was nine hours in length (*cf.* "Secular Changes," *Phil. Trans.*, 1880, p. 881; "Precession," *Phil. Trans.*, 1879, p. 521). Hence, at any rate for any period during which the moon's distance from the earth was more than four times the earth's radius, we may apply the approximate theory for the figure of a nearly spherical body under the action of a distant attracting body to which it always turns the same face, as developed by Laplace and others, and by means of which the above values (1) were obtained. Now it appears that in this theory the quantities $\frac{B-A}{C}$, etc. are proportional to the inverse cube of the distance of the attracting body. In fact if h', h'', h be the semi-axes of the moon in the directions of A, B, C respectively, we have, on any theory as to the constitution of the moon, provided the strata of equal density are concentric similar ellipsoids, the following equations, viz.

$$\frac{B-A}{C} = \frac{h'^2 - h''^2}{h'^2 + h''^2},\quad \frac{C-A}{C} = \frac{h'^2 - h^2}{h'^2 + h''^2},\quad \frac{C-B}{C} = \frac{h''^2 - h^2}{h'^2 + h''^2},$$

which, to the order of approximation required, may be written

$$\frac{B-A}{C}=\frac{h'-h''}{h},\quad \frac{C-A}{C}=\frac{h'-h}{h},\quad \frac{C-B}{C}=\frac{h''-h}{h}\ \ldots(4),$$

and these quantities $\frac{h'-h''}{h}$, etc. we know will be proportional to the first term in the tide-raising potential (the second and higher terms being neglected), *i.e.* to the inverse cube of the distance [*cf.* also Roche, *Mémoires de la Section des Sciences de l'Académie de Montpellier*, I., p. 260, etc., who obtains

$$\frac{h'-h''}{h}=\cdot 00002\ 83,\quad \frac{h'-h}{h}=\cdot 00003\ 78,\quad \frac{h''-h}{h}=\cdot 00000\ 95,$$

which are in substantial agreement with (1), on substituting from (4)].

Let then $a=$ present distance of moon in earth's radii, $a'=$ distance of moon in earth's radii corresponding to its present figure.

Then if λ represent either of the ratios (3) we have

$$\left(\frac{a}{a'}\right)^3=\lambda,$$

or

$$a'=a\left(\frac{1}{\lambda}\right)^{\frac{1}{3}},$$

where $a=60\cdot 3$ (the unit being the earth's radius). The values of a' corresponding to the ratios 10·5, 15·3, 29·9 are respectively 27·5, 24·3, 19·4 (in terms of the same unit).

Allowing for the discrepancy between these numbers, which is very likely due in part to errors of observation, the quantities to be observed being small, we may at least, I think, assert that it is highly probable that the moon cooled to such an extent that its permanent figure was assumed when it was somewhere between a quarter and a half of its present distance from the earth. The supposition that it did so appears to accord well with our present knowledge about the moon; and it at least reconciles the order of magnitude of the observed values with theory: whereas, on the hypothesis that the moon's figure was assumed at its present distance, the observed values are more than ten times greater than they should be.

It may be added that it is only necessary to suppose that the crust solidified at this period; the interior may still have remained molten, so as to permit of internal viscous tides, and consequent elastic oscillations of the crust about its equilibrium position.

THE CONVERSE OF FERMAT'S THEOREM.

By *J. H. Jeans*, Trinity College, Cambridge.

THE problem is to find n, not a prime, so that

$$2^{n-1}-1=0 \ (\text{mod. } n).$$

Let p be any prime factor of n, then

$$2^{n-1}-1=2^{p-1}\left[\{2(2^{p-1}-1)+2\}^{\frac{n}{p}-1}-1\right]+2^{p-1}-1$$

$$=\left[2^{p-1}(2^{\frac{n}{p}-1}-1)\right] \ (\text{mod. } p),$$

therefore we must satisfy the equation $2^{\frac{n}{p}-1}-1=0$ (mod. p) for each prime factor, p, of n. (This is always satisfied if $\frac{n}{p}-1=0$ (mod. $p-1$)).

For the case of two prime factors, we assign to p a series of prime values, factorise $2^{p-1}-1$, and taking each prime factor in turn as a possible value of $\frac{p}{n}$, we apply the test afforded by the above equation.

For values of p from 3 to 31, I have found that the only solutions obtained for n are 341, 1387, 4369, 4681, 10261.

The general case of more than two prime factors, worked out in a similar way for values of p up to 7, gives the single solution $n=645$, which has been noticed by Herr Kossett.

Hence there are only two solutions less than 1000, viz. 341 and 645.

Writing $f(p)$ for $2^{2^p}+1$, $n=f(p)$ is clearly a solution if p is any integer such that $f(p)$ is not prime; and

$$n=f(p).f(q)$$

is another solution if $f(p)$, $f(q)$ are both prime, and $p>q>2^p$. For $2^{f(p)-1}-1=0$ (mod. $f(q)$), and $2^{f(q)-1}-1=0$ (mod. $f(p)$).

The problem has a certain historical interest, since the congruence $2^{n-1}-1=0$ (mod. n) appears to have been known to the Chinese. A paper found among those of the late Sir Thomas Wade, and dating from the time of Confucius, contains the theorem that $2^{n-1}-1=0$ (mod. n) when n is prime, and also states that it does not hold if n is not prime.

It was, presumably, found empirically, and it would in this way be impossible to come upon a case of failure of the second part, seeing that the value of $2^{n-1}-1$ corresponding to the smallest case of failure ($n=341$) consists of 103 figures.

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NOTE ON A FORMULA IN DIFFERENTIATION.

By E. G. *Gallop*.

THIS note refers to an important formula established by Dr. Hobson for the differentiation of $\phi(x_1^2+x_2^2+x_3^2+\dots)$ with respect to the independent variables $x_1, x_2, \dots$ (*Proc. Lond. Math. Soc.*, Vol. XXIV., p. 67; and *Messenger*, Vol. XXIII., p. 117).

The formula states that, if $u=x_1^2+x_2^2+\dots+x_p^2$,

$$f_n\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_p}\right)\phi(x_1^2+x_2^2+\dots+x_p^2)$$

$$=\left[2^n\phi^n(u)+2^{n-2}\phi^{n-1}(u)\nabla^2 + \frac{2^{n-4}}{1.2}\phi^{n-2}(u)\nabla^4+\dots\right]f_n(x_1, x_2, \dots, x_p),$$

where ϕ is an arbitrary function, f_n is a homogeneous integral function of degree n, and

$$\nabla^2=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\dots+\frac{\partial^2}{\partial x_p^2}.$$

This general theorem includes as a particular case the well-known formula for the differentiation of $\phi(x^2)$ with respect to x, and, indeed, merely represents the effect of continued application of the simpler formula. To see this most easily it is convenient to throw the formula into a symbolical form.

Thus we have in the usual way

$$\left(\frac{\partial}{\partial x_1}\right)^r\phi(u)=(2x_1)^r\phi^r(u)+r(r-1)(2x_1)^{r-2}\phi^{r-1}(u)$$

$$+\frac{r(r-1)(r-2)(r-3)}{1.2}(2x_1)^{r-4}\phi^{r-2}(u)+\dots$$

$$=\left[2^r\phi^r(u)+2^{r-2}\phi^{r-1}(u).\frac{\partial^2}{\partial x_1^2}+\frac{2^{r-4}}{1.2}\phi^{r-2}(u)\frac{\partial^4}{\partial x_1^4}+\dots\right]x_1^r$$

$$=(2\delta)^r e^{\delta_1^2/4\delta}.x_1^r\phi(u),$$

where $\delta = d/du$ and operates only on $\phi(u)$, whilst $\delta_1 = \partial/\partial x_1$ and operates only on x_1^r. Hence, also

$$\left(\frac{\partial}{\partial x_2}\right)^s \left(\frac{\partial}{\partial x_1}\right)^r \phi(u) = (2\delta)^s e^{\delta_2^2/4\delta} . x_2^s (2\delta)^r e^{\delta_1^2/4\delta} . x_1^r \phi(u),$$

where $\delta_2 = \partial/\partial x_2$ and operates only on x_2^s. Now δ, δ_1, and δ_2 are obviously commutative. Therefore

$$\left(\frac{\partial}{\partial x_1}\right)^r \left(\frac{\partial}{\partial x_2}\right)^s \phi(u) = (2\delta)^{r+s} e^{(\delta_1^2+\delta_2^2)/4\delta} . x_1^r x_2^s \phi(u).$$

By repeated use of this process we find that

$$\left(\frac{\partial}{\partial x_1}\right)^r \left(\frac{\partial}{\partial x_2}\right)^s \left(\frac{\partial}{\partial x_3}\right)^t \ldots \phi(u) = (2\delta)^n e^{D^2/4\delta} . x_1^r x_2^s x_3^t \ldots \phi(u) \qquad \ldots\ldots\ldots\ldots\ldots (A),$$

where $r+s+t+\ldots = n$, and

$$D^2 = \delta_1^2 + \delta_2^2 + \delta_3^2 + \ldots.$$

This equation is equivalent to Dr. Hobson's formula when the right-hand side is expanded and interpreted, for there is no loss of generality in representing f_n by a single term.

It is also quite easy to establish the theorem by induction, by use of the form (A) last written. For to change r into $r+1$, we operate with $\partial/\partial x_1$ and obtain

$$(2x_1\delta + \delta_1)(2\delta)^n e^{D^2/4\delta} . x_1^r x_2^s x_3^t \ldots \phi(u)$$

$$= (2\delta)^{n+1} \left(x_1 + \frac{\delta_1}{2\delta}\right) e^{D^2/4\delta} . x_1^r x_2^s x_3^t \ldots \phi(u)$$

$$= (2\delta)^{n+1} e^{D^2/4\delta} . x_1^{r+1} x_2^s x_3^t \ldots \phi(u),$$

as is seen at once by writing $e^{D^2/4\delta}$ for $f\left(\frac{\partial}{\partial x_1}\right)$ in the formula

$$f\left(\frac{\partial}{\partial x_1}\right) . x_1 \psi(x_1) = x_1 f\left(\frac{\partial}{\partial x_1}\right) \psi(x_1) + f'\left(\frac{\partial}{\partial x_1}\right) \psi(x_1).$$

ON THE FREQUENCY OF OCCURRENCE OF THE DIGITS IN THE PERIODS OF PURE CIRCULATES.

By *Bion Reynolds, M.A.*, St. John's College, Cambridge.

INTRODUCTORY.

THE chief purport of this paper is to show that the digits in the periods of pure circulating fractions, whether ordinary decimals or other radix-fractions, do not occur, as we might suppose, in a random manner, but according to clear definite laws, and with the utmost possible approach to regularity.

Pure circulates only will be dealt with, and proper fractions only. That is, we shall consider the periods of the radix-fractions corresponding to the vulgar fraction $\frac{M}{N}$, where N is prime to the radix, and M has values from 1 to $N-1$.

It will be convenient first to state the chief laws relating to the length of the period.* Let r denote the radix of notation, and suppose N always prime to r.

LAW 1. If N be a *prime*, and the period of $\frac{1}{N}$ contain n figures, the period of $\frac{M}{N}$ will also contain n figures (M being $2, 3, \ldots, N-1$).

If N be not prime, this law still holds while M is prime to N, but if M is *not* prime to N, the period of $\frac{M}{N}$ may contain only n' figures, where n' is a submultiple of n.

But, whatever N may be (so long as prime to r), the total number of figures involved in *all* the essentially different repeat-series will be $N-1$.†

---

* These are partly taken from an article by Dr. Glaisher, in "Solutions of the Cambridge Senate-House Problems and Riders for 1878" (pp. 8, 9), and from a paper on Circulating Decimals by the same writer in the *Proceedings of the Cambridge Philosophical Society*, Vol. III., pp. 185—206 (1878).

† This will be clearly seen by examples further on, and may be shown to be necessarily the case.

LAW. 2. If N be a prime, the period of $\frac{1}{N}$ will contain $N-1$ figures, or some submultiple of $N-1$.

LAW 3. If N be a prime, and the period of $\frac{1}{N}$ contain n figures, then the period of $\frac{1}{N^2}$ will contain Nn figures, that of $\frac{1}{N^3}$, N^2n figures, and so on.*

LAW 4. If the period of $\frac{1}{N}$, $\frac{1}{N^2}$, ... contain an even number of figures, the second half of these will be complementaries-to-$(r-1)$ of the first half. Thus, in the decimal system, the two halves of such a period, added together, give a row of 9's.

LAW 5. If $\frac{1}{N}$ has a period of n figures, $\frac{1}{P}$ one of p figures, $\frac{1}{Q}$ one of q figures, ..., where $N, P, Q, \ldots$ are primes, then $\frac{1}{N.P.Q\ldots}$ has a period of s figures, where s is the L.C.M. of $n, p, q, \ldots$.

For this class of fractions, Law 4 does not necessarily hold; the two halves of an even period, added together, may give a row of 3's, 6's, or 12's, or some quite irregular result.

LAW 6. As an amplification or combination of Laws 3 and 5, the period of $1/N^{\alpha}P^{\beta}Q^{\gamma}\ldots$ contains s figures, where s is the L.C.M. of $nN^{\alpha-1}$, $pP^{\beta-1}$, $qQ^{\gamma-1}$, ...; if $N, P, Q, \ldots$ be each less than 1000, and not 3 or 487.†

Illustrations of the above laws, in the Decimal system.

1. The period of $\frac{1}{7}$ has 6 figures; that of $\frac{5}{7}$ has the same. The period of $\frac{1}{21}$ has 6 figures; that of $\frac{7}{21}$ has only 1 figure.

2. The period of $\frac{1}{131}$ has 130 figures; that of $\frac{1}{41}$ has $\frac{40}{8}$, or 5 figures; that of $\frac{1}{103}$ has $\frac{102}{3}$, or 34 figures; that of $\frac{1}{37}$ has $\frac{36}{12}$, or 3 figures.

* For a few exceptions to this Law, see Appendix A.

† See Appendix A.

3. The period of $\frac{1}{13}$ has 6 figures; that of $\frac{1}{13^2}$ has 13×6 figures.

4. The period of $\frac{1}{13}$ is 076923; here $0+9=7+2=6+3=9$.

5. The periods of $\frac{1}{11}$ and $\frac{1}{41}$ have 2 and 5 figures respectively; that of $\frac{1}{11 \times 41}$ has 10 figures, viz. $\cdot\dot{0}02217294\dot{9}$. It is seen that the rule of Law 4 does not hold in this case. But again $\frac{1}{7 \times 13} = \cdot\dot{0}1098\dot{9}$, and $\frac{1}{7 \times 13 \times 11} = \cdot\dot{0}0099\dot{9}$, in each of which cases the rule does hold good. Lastly, $\frac{1}{21 \times 11} = \cdot\dot{0}0432\dot{9}$, $\frac{2}{21 \times 11} = \cdot\dot{0}0865\dot{8}$, $\frac{85}{21 \times 11} = \cdot\dot{3}6796\dot{5}$, where the two halves added together give rows of 3's, 6's, 12's. (These rather curious results are capable of very simple explanation).

6. The period of $\frac{1}{11^2 \times 41}$ has $2 \times 11 \times 5$ figures.

Part I. Decimal Fractions.

We now proceed to state the laws of occurrence of digits in the complete sets of repetends belonging to any number N, and begin with consideration of the decimal system, *i.e.* we suppose $r=10$.

(α) If N be of the form $10p+1$, the set of repetends or repeat-series belonging to the fractions $\frac{M}{N}$ will contain each of the ten digits (0, 1, 2, ..., 9) exactly p times.*

Thus

$$\tfrac{1}{61} = \cdot\dot{0}16,\ 393,\ 442,\ 622,\ 950,\ 819,\ 672,\ 131,\ 147,\ 540,$$
$$983,\ 606,\ 557,\ 377,\ 049,\ 180,\ 327,\ 868,\ 852,\ 45\dot{9}.$$

---

* As was remarked under the heading of Law 1, the whole number of figures involved in these series will always be $N-1$, *i.e.* $10p$ in this case.

In this set of 60 figures, each digit occurs 6 times.

So also $\frac{1}{131}$ consists of 130 figures, each digit occurring 13 times.

Next, consider $\frac{1}{31}$, or $\cdot\dot{0}32, 258, 064, 516, 12\dot{9}$. This series serves for only 15 of the 30 proper fractions with denominator 31. The complementary series, that of $\frac{30}{31}$, $\cdot\dot{9}67, 741, 935, 483, 87\dot{0}$, serves for the remaining 15 fractions. Taking the two series together, involving 30 figures, we see that each digit occurs exactly 3 times.

Again, $\frac{1}{101} = \cdot\dot{0}09\dot{9}$, $\frac{2}{101} = \cdot\dot{0}19\dot{8}$, and so on. Taking the 25 such *distinct* repetends together, we find that, in the 100 figures involved, each digit occurs 10 times.*

Now let N be 21. There are altogether 5 distinct repetends; thus

$$\tfrac{1}{21} = \cdot\dot{0}4761\dot{9},\ \tfrac{2}{21} = \cdot\dot{0}9523\dot{8},\ \tfrac{3}{21} = \cdot\dot{1}4285\dot{7},\ \tfrac{7}{21} = \cdot\dot{3},\ \tfrac{14}{21} = \cdot\dot{6}.$$

In the whole set of 20 figures each digit occurs twice.†

(β) If N be of the form $10p + 3$, the whole set of figures involved in all the distinct repetends will be $10p + 2$, and it is thus *impossible* for each digit to occur exactly p times. But the approach to regularity is as great as possible, for while every digit occurs at least p times, the two digits 3 and 6 occur each once in excess, *i.e.* $(p + 1)$ times.

Thus $\frac{1}{23} = \cdot\dot{0}4347826086, 9565217391\dot{3}$,

where 3 and 6 each occur 3 times, instead of twice only, like the others.

The fractions $\frac{M}{33}$ have the following distinct repetends:

03, 06, 09, 12, 15, 18, 24, 27, 3, 36, 39, 45, 48, 57, 6, 69, 78; 32 figures in all, and the 3 and 6 occur each 4 times.

(γ) If N be of the form $10p + 9$, there are $10p + 8$ figures involved in all the repetends. Each digit occurs $p + 1$ times, except 0 and 9, which occur only p times.

---

* This property, applied to primes only, is mentioned in § 12 of Dr. Glaisher's paper on Circulating Decimals, above referred to, but was independently discovered by the present writer.

† It is possible to put the matter in another light. Let $\frac{M}{21}$ be always expressed by a repetend of 6 figures—thus $\frac{7}{21} = \cdot\dot{3}3333\dot{3}$—and let M have all values from 1 to 20. Then, in the 120 figures thus involved, there will be 12 of each digit. This method seems to be more complicated and less scientific than the one hitherto adopted, and so will not be further made use of.

Thus $\frac{1}{19} = \cdot\dot{0}52631578,\ 94736842\dot{1}.$*

(δ) If N be of the form $10p+7$, there will be $10p+6$ figures involved, and the digits 0, 3, 6, 9 will occur p times each, the others $p+1$ times each.

Thus $\frac{1}{17} = \cdot\dot{0}5882352,\ 9411764\dot{7}.$

We have now considered all the cases of N prime to 10, and it is notable that when the digits do not occur with equal frequency, they still occur strictly according to rule, and it is always the equidistant and symmetrically-placed digits 0, 3, 6, 9, or two of them, that occur in a distinctive manner, either by excess or defect. This rule will be re-modelled after we have considered general radix-fractions.

PART II. GENERAL RADIX-FRACTIONS.

It has been noticed that, owing to radix 10 not being a prime number, we had only to consider numbers (N) whose last digit was 1, 3, 7, or 9 as producing pure circulates. By taking a prime number as radix, we can consider a consecutive set of numbers, from $pr+1$ to $pr+r-1$, r being the radix and p any positive integer.

We find that if $N=pr+1$ (so that pr digits are involved) the r digits occur with equal frequency, each p times; if $N=pr+2$, *one* digit, the central one, $\frac{1}{2}(r-1)$, occurs in excess, or *predominates*; if $N=pr+3$, *two* symmetrically-placed digits predominate; if $N=pr+4$, *three* such digits predominate, $\frac{1}{2}(r-1)$ being the middle one of the three. And so on, until, when $N=pr+r-1$, $r-2$ digits predominate, viz. those from 1 to $r-2$. As remarked in Part I., the digits 0 and $(r-1)$ *never* predominate, but are always in defect if any be so. Subject to this rule, it will be found that the predominant digits are always as nearly equidistant as possible. Diagrams will make these points clearer, and will show the beautiful regularity of occurrence of these predominant digits. The following diagrams are constructed for the scales of 7 and 11; the results have been verified by many trials, no exception whatever having been met with.

* It will be found to be a rule in a circulate with radix r that, if *any* of the digits are in a minority, the digits 0 and $r-1$ will be among these.

Septimal Fractions.

Value of the denominator.	No. of figures involved.	Predominant digits.
$7p+1$	$7p$	None
$7p+2$	$7p+1$	3
$7p+3$	$7p+2$	2 4
$7p+4$	$7p+3$	1 3 5
$7p+5$	$7p+4$	1 2 4 5
$7p+6$	$7p+5$	1 2 3 4 5

Undecimal Fractions.

Value of the denominator.	No. of figures involved.	Predominant digits.
$11p+1$	$11p$	None
$11p+2$	$11p+1$	5
$11p+3$	$11p+2$	3 7
$11p+4$	$11p+3$	2 5 8
$11p+5$	$11p+4$	2 4 6 8
$11p+6$	$11p+5$	1 3 5 7 9
$11p+7$	$11p+6$	1 3 4 6 7 9
$11p+8$	$11p+7$	1 2 4 5 6 8 9
$11p+9$	$11p+8$	1 2 3 4 6 7 8 9
$11p+10$	$11p+9$	1 2 3 4 5 6 7 8 9

It will be noticed that in the third section of the diagrams the digits 0 and $r-1$ never appear.

As illustrations, and to assist the reader in verifying the above results, we give a few examples.

(α) *In the* 7-*scale*,

$$\tfrac{1}{11} = \cdot\dot{0}4311,\ 6235\dot{5},$$

where 1, 3, 5 predominate;

$$\tfrac{1}{13} = \cdot\dot{0}35245,\ 63142\dot{1},$$

where 1, 2, 3, 4, 5 predominate.

If N is 19, there are 6 distinct repeat series, viz.

024, 051, 132, 156, 264, 345;

and in this whole set, the digits 1, 2, 4, 5 predominate.

If N is 15, the set of series is as follows:

0316, 0635, 1254, 2, 4;

and, in the set, each of the seven digits occurs twice, exactly.

(β) *In the* 11-*scale*.

$$\tfrac{1}{23} = \cdot\dot{0}5296243390,\ t581486771\dot{t},$$

where each of the 11 digits (t denoting *ten*) occurs twice;

$$\tfrac{1}{13} = \cdot\dot{0}93425,\ t1768\dot{5},$$

where 5 predominates;

$$\tfrac{1}{17} = \cdot\dot{0}7132651,\ t397845\dot{9},$$

where 1, 3, 5, 7, 9 predominate.

If N is 21, the set of series is as follows:

058421, t52689, 163, 947, 37,

and the digits from 1 to 9 predominate, since they occur twice each, while 0 and t occur only once each.

The same general plan holds for all values of r, though the regularity is interrupted, when r is not prime, by the gaps that have to be made in the scheme (viz. in those places where N is not prime to r).

Thus with radix 9,

If $N = 9p + 1$,	no digits predominate,								
„ $= 9p + 2$,	the digit				4				predominates,
„ $= 9p + 4$,	„ digits		2		4		6		predominate,
„ $= 9p + 5$,	„ „	1		3		5		7	„
„ $= 9p + 7$,	„ „	1	2	3		5	6	7	„
„ $= 9p + 8$,	„ „	1	2	3	4	5	6	7	„

These wider investigations show us that we should state the rule for *decimal* fractions with greater consistency, though with less elegance, as follows:

If N ends with 1,	no digit predominates,									
„ „ „ 3,	the digits			3			6			predominate,
„ „ „ 7,	„ „	1	2		4	5		7	8	„
„ „ „ 9,	„ „	1	2	3	4	5	6	7	8	„

These results have been empirically obtained, and the writer does not profess to be able to prove them. Yet he offers them with great confidence, as, after having once discovered what the laws were, he has not met with a single exception. Still, with the striking exception to Law 4 before our eyes (as discussed in Appendix A), it may be rash to positively assert the results to be universally true, until someone has proved them.

Appendix A.

There are two notable exceptions to Law 3, in the Introductory part of this essay, which we will now consider.

First, although $\frac{1}{3} = \cdot\dot{3}$, a period of one figure (in the decimal system), we see that $\frac{1}{3^2} = \cdot\dot{1}$, which is *also* a period of one figure. After this little irregularity, the general principle of the law is maintained, for $\frac{1}{3^3}$ has a 3-period, $\frac{1}{3^4}$ has a 9-period, and so on. This exception to the law is due to the fact

that $3^2 = 10 - 1$, 10 being the radix. If n be *any* integer, and $n^2 = r - 1$ (r the radix of notation), it is easily seen that $\frac{1}{n} = \cdot\dot{n}$, and $\frac{1}{n^2} = \cdot\dot{1}$, so that $\frac{1}{n}$ and $\frac{1}{n^2}$ both have a period of one figure.*

Further, in the scale of 28, the *three* fractions $\frac{1}{3}$, $\frac{1}{3^2}$, $\frac{1}{3^3}$ all have a 1-figure period. After this the general plan of the law is followed, *i.e.* each succeeding power has a period three times the length of the preceding one.

To put the matter another way, the number 3 *does* obey Law 3, except when the radix is of the form $3^p + 1$.†

An exception of quite a different type, discovered by Desmarest,‡ is the number 487. While $\frac{1}{487}$ repeats in 486 places, the repetend itself divides by 487, so that $\frac{1}{487^2}$ *also* repeats in 486 places. The present writer has fully verified these statements, and finds that both series obey Law 4, and the first one ($\frac{1}{487}$) obeys his own law of frequency of digits. To verify the latter result for the 488 distinct repetends belonging to the proper fractions with denominator 487^2, is impracticable, but there is not the least doubt in the mind of the writer that, in these 237168 figures, each digit would occur 23717 times, except 0 and 9, which would occur 23716 times only.

Desmarest further states, and few will care to doubt him, that no *other* number below 1000 possesses the same property as 487 (and 3).

The full series for $\frac{1}{487}$ and $\frac{1}{487^2}$ are given in Dr. Glaisher's pamphlet before mentioned, where it is also stated that the latter repetend does *not* divide by 487, so that $\frac{1}{487^3}$ contains 486×487 figures. It is easy to see, moreover, that $\frac{1}{487}$ itself loses this property in the scales of 486 and 488, and doubtless in many other scales.

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* Thus, in the 26-scale, $\frac{1}{5} = \cdot\dot{5}$, and $\frac{1}{5^2} = \cdot\dot{1}$.

† In the 5-scale, $\frac{1}{3} = \cdot\dot{1}\dot{3}$; $\frac{1}{3^2} = \cdot\dot{0}2342\dot{1}$.

‡ *Théorie des Nombres*, 1852, p. 295.

Law 3 might easily be remodelled so as to make allowance for these occasional irregularities.

APPENDIX B.

Period-lengths with different radices, &c.

Keeping the same number, N, we will consider the length of the period of $\frac{1}{N}$ as depending on the radix.

Take $N=13$. Then the radix-fraction has a repetend of twelve figures, or of n figures (n being *any* submultiple of 12), according to the radix selected.

According as the radix is of the form

$$13p+1,\ 13p+2,\ \ldots\ldots\ldots,\ 13p+12,$$

so the period consists of

1, 12, 3, 6, 4, 12, 12, 4, 3, 6, 12, 2 figures, respectively.

Similarly, if N be 7, with radix $7p+1$, $7p+2$, ..., $7p+6$, we get periods consisting of 1, 3, 6, 3, 6, 2 figures respectively.

The extremes are always 1 and 2, as may very easily be shown to be the inevitable rule.

There is scope for further investigation in this direction.

Lastly, we may trace diagrams to show how the several distinct repetends are distributed among the numerators, in the fractions $\frac{M}{N}$.

To take a very easy case, where N is 13, there are (in the decimal system) two distinct repetends of six figures, each pertaining to six numerators, according to the plan here shown.

Repetends.	Values of the numerator M.											
076923	1		3	4					9	10		12
153846		2			5	6	7	8			11	

This diagram is *laterally* symmetrical, its shape being unaltered if viewed in a looking glass. Others are *diametrically* symmetrical, their shape being unaltered by rotation through 180° in their own plane. These, however, require a certain plan of arrangement of repetends, in order that their symmetry may be brought out.

As an instance of the second class, we get, with $N=13$, and $r=9$, the following scheme:

Repetends.	Values of the numerator M.											
062	1		3						9			
134		2			5	6						
754							7	8			11	
826				4						10		12

ON THE SUPPOSED FIVE-FOLD TRANSITIVE FUNCTION OF 24 ELEMENTS AND 19!÷48 VALUES.

By *G. A. Miller, Ph.D.*

MATHIEU proves the existence of the remarkable five-fold transitive function of 12 elements and 7! values in his *Memoire sur l'étude des fonctions de plusieurs quantities*, which was published in Liouville, 1861, p. 241. In the same memoir he states (p. 274) that he has found a five-fold transitive function of 24 elements which has 19!÷48 values. Twelve years later he published, in the same journal, an article on this function under the heading, *Sur la fonction cinq fois transitive de 24 quantities.* On account of the great interest which such a function would possess, it seemed desirable to make a more complete study of it. It soon appeared that such a function does not exist.* This fact is proved in what follows.

* Cf. Jordan, *Comptes Rendus*, Vol. LXXIX., p. 1149.

If such a function existed there would be a five-fold transitive substitution group of degree 24, and of order 48.20.21.22.23.24. We proceed to prove that such a group (G) cannot be constructed by showing that its existence would lead to contradictions. The subgroups of G which contain all its substitutions that do not involve any one of a given set of n elements will be represented by G_{24-n}.

According to Sylow's theorem G_{23} contains $23k+1$ subgroups of order 23 and

$$48.20.21.22.23 = 23\alpha\,(23k+1),$$

α being a factor of 22. From the equation

$$48.20.21 = 876.23 + 12$$

we observe that $\alpha = 11$. Since α is odd G must be positive. We shall now consider the degree of the β substitutions of order 3 that occur in G_{19}. This degreee cannot be 6, 9, 12, or 15, for none of the numbers

$$\frac{20.\beta}{14},\ \frac{20.\beta}{11},\ \frac{20.21.\beta}{8.9},\ \frac{20.21.22.23.24.\beta}{5.6.7.8.9}$$

is an integer, β being a power of 2. Hence the required degree is 18, and the class of G cannot be less than 8, $ab.cd.ef$ being negative.

We proceed to determine the value of β. Let b be the number of the substitutions of G_{19} that transform one of its subgroups of order 3 into itself. It is evident that $b\beta = 96$. The number of the substitutions of G that transform the same subgroup into itself is 6! into b. Since this subgroup has six systems of intransitivity, and the number of the substitutions of G that transform it into itself without interchanging any of these six systems is $2^\gamma.3^\delta$, $\gamma = 0$ or 1, we have the equation $b.6! = 2^\gamma.3^\delta$, $6! \div \theta$, or $b = 2^\gamma.3^\delta \div \theta$.

Since a group of order 48 contains 1, 4, or 16 subgroups of order 3, b is one of the three numbers 48, 12, 3, We observed above that b is not divisible by 4. Hence we see that $b = 3$ and $\beta = 32$, *i.e.* G_{19}, as an operation group, is one of the two groups of order 48 that contain 16 subgroups of order 3 and one of order 16. As a substitution group it contains 32 substitutions of order 3 and degree 18. We proceed to determine the form of the substitution of the subgroup of order 16.

If α of these substitutions were of degree 8, we would have that $\frac{20.\alpha}{12}$ and $\frac{20.21.\alpha}{12.13}$ are integers. This is impossible since α is not divisible by 39. Hence the class of G cannot be less than 12. If β of the given substitutions were of degree 12 the two numbers $\frac{20.\beta}{8}$, $\frac{20.21.\beta}{9}$ would be integers, so that β would be divisible by 6. This could only be possible when G_{19} contains 15 substitutions of order 2. If G_{19} contains substitutions of order 4, it must contain 12 of this order and 3 of order 2.* All of these must be of degree 16 since no positive substitutions of order 2 could be of degree 14 or 18, and the number of the substitutions of degree $16 \equiv 0 \pmod 3$.

In this case G would contain $\frac{3.20.21.22.23.24}{4.5.6.7.8}$ conjugate substitutions of order 2 and degree 16. Each of these would therefore be transformed into itself by a subgroup of order $\frac{8.8!}{3}$. Since such a substitution contains only 8 systems of intransitivity, and since G contains only 2 substitutions that transform each of these systems into itself, the order of the given subgroup should divide 2.8!. Hence such a group cannot exist. It remains only to consider the cases when the subgroup of order 16 which is contained in G_{19} contains no substitution whose order exceeds 2. It has been proved that all of these substitutions are of degree 12 or 16 and that the number of those of degree 12, if such occur, must be divisible by 6.

This number could not be 6, for the average number of elements in all the substitutions of G_{19} must be an integer, but $(6.12 + 9.16 + 32.18) \div 48$ is not an integer. If it were 12 the number of substitutions of this degree in G would be

$$\frac{12.20.21.22.23.24}{8.9.10.11.12}.$$

If all of them were conjugate each one would be transformed into itself by a subgroup of order 4.8.9.10.11.12. This is clearly impossible since such a substitution would contain only 6 systems of intransitivity, and no substitution of order 11 could transform all of these systems into themselves. Similarly, we see that $\frac{1}{4}$, $\frac{1}{2}$, or $\frac{3}{4}$ of them could not be conjugate. Hence such a group is impossible.

* Cf. *Quarterly Journal of Mathematics*, Vol. XXVIII., p. 271.

It remains to consider the case when G_{19} contains 15 substitutions of degree 16 and of order 2. G would contain

$$\frac{15.20.21.22.23.24}{4.5.6.7.8}$$

such substitutions and the order of the largest subgroup which transformed one of them into itself would divide $2.8\,!$. If all of them were conjugate each one would be transformed into itself by $\frac{8.8\,!}{15}$ substitutions. As this number does not divide $2.8\,!$ the given substitutions would have to be conjugate in sets, each set containing $\frac{\alpha}{5}$ $(\alpha = 1, 2, 3, 4)$ of the entire number. If $\alpha = 1$, 2, or 3 each one of the set would be transformed into itself by $\frac{8.8\,!}{3}$, $\frac{8.8\,!}{6}$, $\frac{8.8\,!}{9}$ substitutions. As none of these numbers is a factor of $2.8\,!$ there can be no system which includes just $\frac{1}{5}$, $\frac{2}{5}$, or $\frac{3}{5}$ of all these substitutions. If there were a system which included just $\frac{4}{5}$ of all these substitutions there would also be a system which included just $\frac{1}{5}$ of them. As this is impossible the given G cannot be constructed, and the given five-fold transitive function can therefore not exist.

Cornell University.

Correction to Prof. Forsyth's Paper, p. 117.

In the paper, pp. 99–118 of the present volume, one modification (on p. 117) is required. The quantity ζ is a function of u and η such that

$$\begin{vmatrix} u, & \eta, & \zeta \\ p, & p', & p'' \\ q, & q', & q'' \end{vmatrix} = 0;$$

consequently the arbitrary function in § 14 involves only the two arguments u and η. This oversight was pointed out to me by Prof. M. J. M. Hill.

A. R. F.

END OF VOL. XXVII.

CONTENTS.

Papers for insertion should be sent to Dr. Glaisher, Trinity College, Cambridge, or to Messrs. Metcalfe and Co. Limited, Printing Office, Trinity Street, Cambridge.